Control of interfacial instabilities using flow geometry

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Abstract

This supplementary material includes a more detailed derivation of the dispersion relation obtained from linear stability analysis. In addition, supplementary results and movies of the experiments are attached to visualize the change in interfacial behaviour due to the presence of the depth gradient in the Hele-Shaw cell.
THEORETICAL ANALYSIS

This section provides a more general derivation of the theoretical result reported in the main article. The Hele-Shaw cell, of depth $h(x) = h_0 + \alpha x$ and width $W$, is either slowly expanding or contracting [1, 2]; this geometric change sets the sign of the gradient $\alpha$. Fluid 1, of density $\rho_1$ and viscosity $\mu_1$, is displaced by the invading fluid 2 of density $\rho_2$ and viscosity $\mu_2$. We can write $\mu_j = \rho_j^{-1} \mu_1$, where $\lambda = \mu_2/\mu_1$ is the viscosity ratio of the fluids and index $j$ refers to the fluids ($j = 1, 2$). The fluids are immiscible with surface tension $\gamma$ and gravity is included. The base state of the system is a uniform interface at $x = 0$ and advancing uniformly at a velocity $U$; see Fig. 1. In the presence of a depth gradient along the flow direction, a flat interface propagates with $U(t)$. We note that we can examine the linear stability of a flat interface at a given location, i.e. with a given velocity $U$, since the governing equations of viscous flows are quasi-steady.

FIG. 1. Schematic of a Hele-Shaw cell with a depth gradient $|\alpha| \ll 1$. The cell is either expanding ($\alpha > 0$) or contracting ($\alpha < 0$). Since $h(x) \ll W$ and $h(x) \ll L$, only the depth-averaged dynamics of the system are analyzed.

Since $h(x) \ll W$ and $h(x) \ll L$, the governing equations of the system are the depth-averaged Darcy’s law, and, the mass conservation equation modified to account for the depth gradient:

\begin{align}
\mathbf{u}_j &= -\frac{h^2}{12 \mu_j} \left( \nabla p_j - \rho_j \mathbf{g} \right), \quad (1) \\
\nabla \cdot (h \mathbf{u}_j) &= 0, \quad (2)
\end{align}

where $p_j$ and $\mathbf{u}_j$ are, respectively, the depth-averaged pressure and velocity of phase $j$; $\mathbf{g}$ is the gravitational acceleration. The system is rendered dimensionless using convenient characteristic scales: $x_c = h_0/3$, $h_c = h_0$, $y_c = W$, $u_c = U$ and $p_c = 4 \mu_1 U / h_0$. Henceforth, all variables are nondimensional. To avoid any confusion, note that all the variables that appear in the main article are dimensional with the exception of the wave number $k$, which is made dimensionless for convenience.

Substituting equation (1) into (2), and neglecting terms of $O(\alpha^2)$ since $|\alpha| \ll 1$, yields

\begin{align}
\frac{\partial^2 p_j}{\partial x^2} + \frac{\alpha^2}{9} \frac{\partial^2 p_j}{\partial y^2} + \alpha \left( \frac{\partial p_j}{\partial x} - \frac{\rho_j g x h_0^2}{12 \mu_1 U} \right) &= 0, \quad (3)
\end{align}
where \( a = h_0/W \) is the characteristic aspect ratio of the cell and \( g_x \) is the component of gravity in the \( x \)-direction, i.e. the flow direction.

The interface is perturbed with a small amplitude normal mode \( \varepsilon(y, t) = \varepsilon_0 e^{i k y + \delta t} \ll 1 \), where \( k \) and \( \delta \), respectively, are the dimensionless wave number and the dimensionless growth rate. The wave number \( k \) is taken to be positive without loss of generality. Accounting for these periodic interfacial perturbations, the general solution of equation (3) is written as

\[
p_j(x, y, t) = \frac{\rho_j g_x h_0^2}{12 \mu_1 U} x + f_j(x) + q_{jk}(x) \varepsilon(y, t),
\]

where \( f_j(x) \) represents the pressure drop due to a uniformly displaced interface (base state). Also, \( q_{jk}(x) \) is the contribution of the interfacial disturbance whose wave number is \( k \). Consistent with these definitions, we demand that \( \frac{df_j}{dx} \bigg|_{x=0} = -\lambda^{j-1} \) (Darcy’s law) and that \( q_{jk}(x) \) must vanish away from the interface; \( \lim_{x \to -\infty} q_{1k}(x) = \lim_{x \to -\infty} q_{2k}(x) = 0 \). The reference pressure is arbitrary. Substituting equation (4) into (3), and applying the previous conditions, we can solve for \( f_j(x) \) and \( q_{jk}(x) \). Equation (4) becomes

\[
p_j = \frac{\rho_j g_x h_0^2}{12 \mu_1 U} x + \frac{\lambda^{j-1}}{\alpha} e^{-\alpha x} + \sum_k b_{jk} e^{m_{jk} x} \varepsilon,
\]

where \( b_{jk} \) is a constant to be determined.

As the unperturbed interface is located at \( x = 0 \), the position of the perturbed interface is given by \( x_{int}(y, t) = \varepsilon = \varepsilon_0 e^{i k y + \delta t} \). The kinematic boundary condition, which imposes continuity of velocities at the interface, is now written in dimensionless form as

\[
\frac{\partial x_{int}}{\partial t} = \frac{u_{jx}}{3} \bigg|_{x=x_{int}} + \frac{\partial x_{int}}{\partial y} u_{jjy} \bigg|_{x=x_{int}} - 1,
\]

where the velocities \( u_{jx} \) and \( u_{jjy} \) are given by Darcy’s law in equation (1),

\[
u_{jx} = -\frac{h^2}{\lambda^{j-1}} \left( \frac{\partial p_j}{\partial x} - \frac{\rho_j g_x h_0^2}{12 \mu_1 U} \right) \quad \text{and} \quad u_{jjy} = -\frac{a}{3} \frac{h^2}{\lambda^{j-1}} \frac{\partial p_j}{\partial y}.
\]

Here, \( h(x) = 1 + \frac{q}{3} x \) and the \(-1\) in equation (7) corresponds to subtracting the base state. In addition, equation (7) assumes a sharp interface and hence neglects the effect wetting films. In our experimental range, wetting films are estimated to be less than 3% of the depth of the Hele-Shaw cell and so can be neglected [3]. Equations (8) and (5) are substituted into the kinematic boundary condition given in (7). Neglecting terms of \( O(\varepsilon) \) and \( O(\varepsilon^2) \) and using orthogonality, we get

\[
b_{jk} = -\frac{\lambda^{j-1}}{m_{jk}} \tilde{\sigma}.
\]

The pressure drop at the interface, described by the Young-Laplace equation, accounts for the curvature due to the depth as well as the lateral curvature. In dimensionless form, the Young–Laplace equation is

\[
p_2 - p_1 = \frac{3}{Ca} \left( \frac{2 \cos \theta_c}{h(x)} - \frac{a^2}{3} \frac{\partial^2 x_{int}}{\partial y^2} \right) \quad \text{at} \quad x = x_{int}(y, t),
\]
where the capillary number $Ca = 12\mu_1 U/\gamma$ is the ratio of viscous forces to surface tension forces. The contact angle at the interface $\theta_c$ is taken to be 0 if fluid 1, i.e. the displaced fluid, is perfectly wetting. The influence of the wetting film on the interfacial pressure drop is small for the range of $Ca$ in our experiments [4, 5].

The dispersion relation $\tilde{\sigma}(k)$ is found by first substituting equation (5) into (10). Equations (6) and (9) are then substituted in the resulting expression, which is linearized in $\varepsilon$. Using orthogonality and in the limit where $|\alpha| \ll 1$ and $Ca \ll 1$, we obtain

$$\frac{3}{a} (1 + \lambda) \tilde{\sigma} = \left( 1 - \lambda + G + \frac{2\alpha \cos \theta_c}{Ca} \right) k - \frac{a^2}{Ca} k^3,$$

(11)

where $G = \frac{(\rho_2 - \rho_1) g h_0^2}{12 \mu_1 U}$ is the ratio of gravitational to viscous forces. In the absence of gravity i.e. $G = 0$, and with $\tilde{\sigma} = \frac{a}{u_c} \sigma = \frac{h_0}{3U} \sigma$, the dispersion relation reported in the article is recovered from equation (11). The interface is stable if a mode of any given wave number $k$ decays, i.e. $\tilde{\sigma} < 0$ for all $k > 0$. The latter is equivalent to stating that the interface is stable if

$$1 - \lambda + G + \frac{2\alpha \cos \theta_c}{Ca} \leq 0.$$

(12)

The threshold or critical capillary number $Ca_c$ is then given by:

$$Ca_c = \frac{2\alpha \cos \theta_c}{\lambda - G - 1}.$$

(13)


SUPPLEMENTARY GRAPHS AND MOVIES

The supplementary movie displays the effect of the gradient on the interfacial behaviour. In particular, the video shows two experiments where air displaces mineral oil ($\mu_1 = 25$ cP, $\gamma = 29$ mN/m) in a Hele-Shaw cell, which is 5.1 cm in width and has a negative depth gradient $\alpha = -2.7 \times 10^{-3}$. In the first experiment, which is conducted at a higher capillary number $Ca = 0.016$, the interface is unstable since a uniform interface develops into a single finger, which is wider than half of the cell width. In the second experiment, which is conducted at a lower capillary number $Ca = 0.006$, the interface is stable as it propagates uniformly without forming a finger.

Assuming a Hele-Shaw cell to have a constant gap, an air finger would displace a viscous fluid at a constant speed. In the presence of a negative gradient, the cross-sectional area
of the cell decreases slowly in the flow direction and that can cause a slight increase in the speed of the finger. Furthermore, as the cell is becoming shallower in the flow direction, the nose of the finger experiences more surface tension forces than its sides. Consequently, the sides of the finger exhibit relaxations in the lateral direction of the cell. This effect leads to fingers that are wider (Fig. 2) than the well-known half-width fingers and, in addition, can result in slight non-monotonic variations in the speed of the finger nose.

FIG. 2. Wider air fingers in the presence of a depth gradient. The width of air fingers penetrating through mineral oil (viscosity $\mu_1 = 25$ cP, surface tension $\gamma = 29$ mN/m) in converging Hele-Shaw cell, $\alpha = -2.7 \times 10^{-3}$. The actual finger width is scaled with the cell width $W$. As opposed to classical viscous fingering experiments, the air fingers in experiments with unstable behaviour (○) occupy more than half of the Hele-Shaw cell width. In experiments with stable behaviour (●), the interface advances uniformly and the "finger" width is equal to the cell width. As in Fig. 2d, (●) is the experimental crossover, where the theoretical prediction (....) is obtained using the derived stability criterion.