Para-hydrogen raser delivers sub-millihertz resolution in nuclear magnetic resonance

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I Maser equations and key dynamical parameters

In the rotating frame the motion of the transverse and longitudinal magnetization $M_T(t)$ and $M_z(t)$ of a raser operating at a single frequency can be described by a set of nonlinear coupled differential equations:

$$\frac{d}{dt}M_z(t) = - \frac{M_T^2(t)}{\tau_{rd}M_0} + \frac{1}{\tau_p} \{M_0 - M_z(t)\} - \frac{M_z(t)}{T_1}, \quad (3)$$

$$\frac{d}{dt}M_T(t) = M_T \left\{ \frac{M_z(t)}{\tau_{rd}M_0} - \frac{1}{T_2^*} \right\}. \quad (4)$$

Numerical simulations (see Fig. S1c) show that the characteristic motion of $M_T(t)$ and $M_z(t)$ are determined by four constants, namely the radiation damping rate $1/\tau_{rd}$, the pumping rate $1/\tau_p$, the overall transverse relaxation rate $1/T_2^* = 1/T_2 + 1/\tau_p$ and the longitudinal relaxation rate $1/T_1$.

Assuming $M_z(0) = 0$ and $M_T(0) \approx 0$ (quantum fluctuations) at $t = 0$ and neglecting any back action of the detection coil, equation (3) predicts an initial exponential growth of $M_z(t)$ with the pumping rate $1/\tau_p$ towards the equilibrium magnetization $M_0$. Note that $M_T$ is always positive but $M_0$ is negative (population inversion) and $M_z(t)$ starts to grow negatively at first. When the negative magnetization exceeds the threshold $M_z(t) = M_0\tau_{rd}/T_2^*$ the braced term in equation (4) switches from negative to positive sign, so $dM_T(t)/dt > 0$ and raser oscillation starts. From equation equation (4) follows that $1/\tau_{rd} > 1/T_2^*$ ($\tau_{rd} < T_2^*$) is indispensable for raser activity. An oscillatory motion of $M_z(t)$ results from equations (3) and (4). This oscillation is damped until $dM_z/dt = 0$ and the equilibrium value $M_z^e = M_0\tau_{rd}/T_2^*$ and a constant $M_T^e = M_0(\tau_{rd}/\gamma_0(1 - \tau_p/T_2^*))^{1/2}$ is reached. The superscript “e” stands for equilibrium. In the laboratory frame $M_T^e$ oscillates around the $B_0$ field with the Larmor frequency $\nu_0 = \gamma B_0/2\pi$ and the steady state motion can be described by a rotating cone with a constant $M_T^e$, which points in opposite direction with respect to the direction of $B_0$. Thus $M_T^e$ represents a population inversion or a state with negative spin temperature. The start-up spikes of the initial relaxation oscillations, omitting $T_1$ relaxation rate, are damped out with a late time damping rate $1/(2\tau_p)$ and are separated by a characteristic late time period given by

$$T_L = 4\pi \tau_p/\sqrt{8\tau_p(1/\tau_{rd} - 1/T_2^*)} - 1. \quad (5)$$

The values for $1/\tau_p$ and $T_L$ can be determined from the late time decay and the separation between subsequent raser pulses. $1/T_2$ can be measured from the free induction decay of the corresponding NMR experiment. From Figs. 1b,c we get $\tau_{p,py} = 20$ s, $T_{2,py} = 0.7$ s,
The radiation damping time $\tau_{rd}$ can be estimated from equation (5) given the measured values for $T_L$, $T^*_2$ and $\tau_p$. This results in $\tau_{rd,py} = 0.016 \text{s}$ for pyridine and $\tau_{rd,ac} = 0.18 \text{s}$ for acetonitrile. The proton polarization $P_H$ for the experiments shown in Figs. 1b,c can be estimated from equation (2) using the known parameters $\eta = 0.25$, $Q = 300$, and the spin number densities $n_{s,py} = 7.5 \cdot 10^{19} \text{ cm}^{-3}$, $n_{s,ac} = 3.4 \cdot 10^{20} \text{ cm}^{-3}$. We obtain $P_{H,py} = -3.8 \cdot 10^{-3}$ and $P_{H,ac} = -7.3 \cdot 10^{-5}$. So acetonitrile is about 52 times less polarized compared to pyridine. To conclude equations (3–5) together with the key parameters correctly describe the dynamics of the raser signals presented here pumped to a state characterized by a negative spin temperature (or by a single-spin order $\sim \alpha I_z$, $\alpha < 0$) and oscillating at one single frequency.

II Distant dipolar fields

Distant dipolar fields are important since they influence the motion, the observed frequency as well as the precision of frequency measurements of molecular rasers. Desvaux\textsuperscript{30} describes various non-linear effects in liquid state NMR systems based on distant dipolar fields. The average dipolar field for $^1$H-spins is given by $B_{dip} = \mu_0 M_0 = 0.5 \mu_0 n_s \gamma_H \hbar P_H$, where $n_s$ is the spin number density and $P_H$ is the proton polarization. For the raser experiment with pyridine $n_{s,py} = 7.5 \cdot 10^{19} \text{ cm}^{-3}$ and $P_{H,py} = 3.8 \cdot 10^{-3}$ so the calculated dipolar field is $B_{dip} = 5.0 \text{ nT}$. This corresponds to a $^1$H frequency shift of $\Delta \nu_{dip} = 214 \text{ mHz}$. We verified the dipolar shift experimentally for the pyridine raser signal shown in Fig. 1b, an expansion of which is shown in Fig. S1a. In particular the first raser burst around $t = 0 \text{s}$ is accompanied by a sign change in the corresponding $M_z$ magnetization (Fig. S1c) and therefore undergoes roughly a frequency shift $\Delta \nu_{\text{meas}}$. The highly resolved Fourier transformed spectra before and after $t = 0$ yield a measured frequency difference of $\Delta \nu_{\text{meas}} = 224 \text{ mHz}$ (Fig. S1b), which is close to the calculated $\Delta \nu_{dip}$. 

$T_{L,py} = 2.6 \text{s}$ for pyridine and $\tau_{p,ac} = 10 \text{s}$, $T^*_2,ac = 0.6 \text{s}$, $T_{L,ac} = 8.8 \text{s}$ for acetonitrile.
**Fig. S1:** Distant dipolar field of a $^1$H pyridine maser with single spin-order.

**a** Time expansion of Fig. 1b around first maser burst at $t = 0\text{s}$ showing a starting pyridine maser.  

**b** Fourier transformed spectra before (red) and after (blue) the maximum of the first maser burst. The difference between the two maxima is given by $|\Delta\nu_{\text{meas}}| = 224\text{ mHz}$, which corresponds to a distant dipolar field of $B_{\text{dip}} = -5\text{ nT}$. The negative offset frequency axis comes from the choice of the reference frequency at 166.66 kHz.

**c** Simulation of $M_T$ and $M_z$ versus time during the first maser burst based on equations (3) and (4). Simulation parameters: $\tau_p = 20\text{s}$, $T_1 = 2.7\text{s}$, $T_2 = 0.7\text{s}$ and $\tau_{rd} = 0.016\text{s}$. $M_z$ changes sign shortly after $M_T$ reaches its maximum. The red and blue bars correspond to the time intervals highlighted in **a**, which were in turn used to determine $|\Delta\nu_{\text{meas}}|$. 
III Quantum picture of a raser based on two-spin order

The vector cone representation of raser operation (Fig. 2c) is an intuitive way to explain the main differences between a raser pumped into a state characterized by a negative spin temperature (population inversion) and by a two-spin state. Fig. S2 shows the corresponding energy level diagram of a hetero-nuclear $J$-coupled two-spin system $I - S$. The four energy levels are indicated by $E_i$ with $i = 1..4$. In the case of a population inversion (Fig. S2a) the highest energy state $E_4$ is the most populated, and the lower level’s population descends in order. Consequently all four possible dipole transitions with frequencies $\nu_I \pm J/2$, $\nu_S \pm J/2$ have an individual population inversion that can be stimulated to emit radiation (raser). In contrast the two-spin ordered state (Fig. S2b) can be characterized by an increased population of the levels $E_2$ and $E_3$ while $E_3$, $E_4$ are depopulated (quadrupolar distribution). Now only the lower two transitions at frequencies $\nu_I - J/2$, $\nu_S - J/2$ can be raser active while the upper transitions at $\nu_I + J/2$, $\nu_S + J/2$ are raser inactive, since the corresponding energy levels have no population inversion. All four lines can be observed in a standard NMR experiment after pulse excitation as either $I$- or $S$-spin spectrum with anti-symmetric structure (Fig. S3 a-d). Inversion of a state with population inversion or negative spin temperature (Fig. S2a) inhibits raser activity, while inversion of the two-spin state (Fig. S2b) shifts the raser active modes from $\nu_{I,S} - J/2$, to $\nu_{I,S} + J/2$. This fundamental difference distinguishes the two-spin ordered raser from all other known rasers and masers. With regard to the SABRE mechanism this means that raser action is independent of the individual signs of the quantum mechanical coefficients characterizing the SABRE polarization transfer.
Fig. S2: Raser of a $J$-coupled I-S spin system based on single- and two-spin order.

**a, b.** Energy level diagram (left) and schematic spectrum of raser active modes (right) of a hetero-nuclear $J$-coupled two-spin 1/2 system I-S. The energy levels are given by

\[
\begin{align*}
E_1/h &= -\nu_I/2 - \nu_S/2 + J/4 \\
E_2/h &= -\nu_I/2 + \nu_S/2 - J/4 \\
E_3/h &= \nu_I/2 - \nu_S/2 - J/4 \\
E_4/h &= \nu_I/2 + \nu_S/2 + J/4
\end{align*}
\]

where $\nu_I = \gamma_I B_0/(2\pi)$ and $\nu_S = \gamma_S B_0/(2\pi)$. Circles indicate the population of each level. The transition frequencies of each raser active modes are indicated by arrows. For the case of a population inversion ($\alpha(I_z + S_z), \alpha < 0$) both I spin and S-spin transitions are raser active. If the initial state is a two-spin ordered state ($\beta(I_z, S_z), \beta < 0$) only one raser transition for either I- or S-spin can be observed.
Fig. S3: $^1$H and $^{15}$N signals from $^{15}$N-acetonitrile.

a. $^1$H raser signal of $^{15}$N-acetonitrile ① measured at $B_0 = 0.977$ mT (41.6 kHz) at $t < 0$ s. After 90° pulse excitation at $t = 0$ a long FID appears ②. b, Fourier spectrum of panel a. ① (top), of ② (middle). The simulated $^1$H spectrum (bottom) results from pure two-spin order. Note the raser line which appears only on the left side of the spectrum. c, Measured $^{15}$N (top) and simulated $^{15}$N FID (bottom) of $^{15}$N-acetonitrile after 90° pulse excitation ($B_0 = 9.65$ mT, 41.6 kHz). d, Measured (top) and simulated (bottom) $^{15}$N spectrum from panel c.
IV Distant dipolar field immunity of the two-spin based raser

A further important difference exists between a two-spin ordered raser and a raser based on population inversion. The raser based on two spin order, which applies to our $^{15}$N labelled substrates at low field, is nearly immune against distant dipolar fields (DDF). This can be graphically illustrated by two antiparallel magnetizations ($M_+^z$ and $M_-^z$) which cancel out all DDF effects. A slight imperfection arises from the reduction in the $M_z$ magnitude due to the raser action (see Fig. 2c) where $|M_{	ext{e-}}^z| \lesssim |M_{	ext{e+}}^z|$. For the case of $^{15}$N labelled pyridine raser the magnitude of the $^1$H–$^{15}$N two-spin order is at least one order of magnitude smaller than the dipolar magnetization (single-spin order) of pure pyridine. Since we measured a maximum dipolar shift effect of $\approx 224$ mHz for the pyridine raser (section II) we can estimate the constant dipolar shift for the two-spin based $^{15}$N-pyridine raser, taking into account the non-perfect cancelation $M_{z-}^e$ and $M_{z+}^e$, to be in the order of a few mHz. This is small compared to the maximum frequency fluctuations of 400 mHz as reported in Fig. 4a. Therefore the observed frequency variations are caused mainly by ambient magnetic field fluctuations or by the temporal instability of the $B_0$ current source. Thus the self-compensating two-spin ordered raser is suitable as a reliable magnetic field sensor. When measuring $J$-coupling constants using frequency differences in the spectra external magnetic fields as well as slight internal DDF fluctuations in the mHz regime cancel out, leading to a precision far below the milli-Hertz regime. Finally the absence of DDF explains why we could not observe non-linear effects associated to DDF, such as dipolar echoes, spectral clustering, spin turbulence, in our $^{15}$N-pyridine raser experiments.

In summary the immunity against DDF, the absence of non-linear effects and the fact that the bound complex is NMR and raser silent (see section V) distinguishes the raser based on two-spin order from rasers and masers based on population inversion. Therefore the hetero-nuclear two-spin ordered raser opens the door for high-precision measurements of molecular coupling parameters, magnetic fields and rotational motions. This is not only important for material sciences and sensor-technology but also for fundamental physics, for example for the two nuclei $^3$He–$^{129}$Xe Zeeman raser, which is plagued by distant dipolar field effects.
Simulation of SABRE NMR spectra

Valuable information about the spin state and the possible maser active frequencies for $^{15}$N labeled pyridine and acetonitrile can be gained from measurements and simulations of the corresponding $^1$H and $^{15}$N NMR spectra. For $^{15}$N-pyridine we define the set of one $^{15}$N spin at position 1 (S-spin) and five proton spin-vector operators (I-spins) at positions 2–6 as $\{S, I_2, I_3, I_4, I_5, I_6\}$ (Fig. 3c). Correspondingly for $^{15}$N-acetonitrile we have $\{S, I_2, I_3, I_4\}$ (Fig. 2a). The Hamilton operator for $^{15}$N-pyridine can be expressed as

$$H_{py} = \omega_1 \sum_{j=2}^{6} I_{jz} + \omega_S S_z + 2\pi \left\{ J_{12} S \cdot (I_2 + I_6) + J_{13} S \cdot (I_3 + I_5) + J_{14} S \cdot I_4 + J_{23} (I_2 \cdot I_3 + I_5 \cdot I_6) + J_{25} (I_2 \cdot I_5 + I_3 \cdot I_6) + J_{34} I_4 \cdot (I_3 + I_5) + J_{24} I_4 \cdot (I_2 + I_6) + J_{35} I_3 \cdot I_5 + J_{26} I_2 \cdot I_6 \right\}$$

(6)

and for $^{15}$N-acetonitrile it is given by

$$H_{ac} = \omega_1 \sum_{i=2}^{4} I_{i z} + \omega_S S_z + 2\pi J_{NH} S \cdot (I_2 + I_3 + I_4) + 2\pi J_{HH} (I_2 \cdot I_3 + I_2 \cdot I_4 + I_3 \cdot I_4)$$

(7)

The Larmor frequency of the $^1$H ($^{15}$N) spin in equations (6,7) is given by $\omega_1 = \gamma_1 B_0$ ($\omega_S = \gamma_{15N} B_0$). For pyridine the influence of the chemical shift of the two ortho (I$_2$, I$_6$, $\delta_0 = 8.43$ ppm), meta-protons (I$_3$, I$_5$, $\delta_m = 7.34$ ppm) and the para proton (I$_4$, $\delta_p = 7.75$ ppm) can be neglected, since their frequency differences at $B_0 \approx 1$ mT are very small ($< 0.04$ Hz).

For $^{15}$N-pyridine at 1 mT the difference in frequency between the ortho, meta and para protons is dominated by the difference in the corresponding hetero-nuclear J-coupling constants, which is in the same order of magnitude compared to the homo-nuclear coupling constants. Thus the condition for strong coupling is fulfilled. Therefore a calculation of the frequency difference between possible J-coupled spin states needs to take into account all off-diagonal elements in the scalar products in equations (6,7). We found the calculated and measured frequency differences in the strong coupling regime are very close to the J-coupling constants measured in high field. The $^1$H–$^{15}$N J-coupling constants of $^{15}$N-pyridine measured at high field are$^{20}$: $\{J_{12}, J_{13}, J_{14}\} = \{-10.06, -1.56, \pm0.18\}$ Hz, and the homo-nuclear $^1$H–$^1$H J-coupling constants are: $\{J_{23}, J_{24}, J_{25}, J_{26}, J_{35}, J_{34}\} = \{4.97, 1.81, 0.90, -0.16, 1.38, 7.83\}$ Hz. For $^{15}$N-acetonitrile we find $J_{NH} = 1.72$ Hz.
If acetonitrile or pyridine without $^{15}\text{N}$ is attached to the Ir-complex the singlet state of the bound para-hydrogen molecule is converted, depending on the field, into observable single spin order $\sim -I_{z}$ and/or into multiple spin order by the SABRE mechanism$^{14,15}$. For example the $^1\text{H}$ single spin order of pyridine at $B_0 = 6.6 \text{ mT}$ at $t = 0$ can be expressed as $\rho_{py}(0) = \alpha(I_{2z} + I_{5z}) + m(I_{3z} + I_{5z}) + pl_{4z}$. The three parameters $\alpha = -0.072$, $m = -0.038$ and $p = -0.081$ represent the amount of spin polarization of the ortho, meta and para protons$^{14,29}$. This initial state implies that at 6.6 mT all protons of pyridine are characterized by a population inversion and are raser active provided the threshold condition given by equation (1) is fulfilled.

For $^{15}\text{N}$ labelled pyridine and acetonitrile the initial density matrices $\rho_{py}(0)$ and $\rho_{ac}(0)$ at fields $B_0$ below a few mT have a different structure. Especially two- and higher spin-order can be generated in the SABRE complex. Let us propose a mechanism which creates mainly hetero-nuclear $I - S$ two-spin order at the target molecule. In a simple case two molecules $I - S$ and $I^* - S^*$ are bound to the equatorial plane in the iridium-(Ir) complex, where $I, I^*$ represent two $^1\text{H}$-spins and $S, S^*$ represent the $^{15}\text{N}$ spins on both molecules. On the other side of the equatorial plane of the Ir-complex the two protons from para-hydrogen are bound to the Ir. These two protons with their corresponding denominators $I_P$ and $I_{P^*}$ are initially in a singlet state $|S^{P,P^*}\rangle$. The topology of this six spin system is $I_P, I_{P^*}, S, S^*, I, I^*$. If both $S (^{15}\text{N})$ spins are either up or down ($m_S = m_{S^*} = \pm 1/2$) an efficient transfer of singlet spin-order from para-hydrogen to the intermolecular pair of protons $I-I^*$ of both molecules exists at a specified low magnetic field ($0.01 \text{ mT} < B_0 < 4 \text{ mT}$). This so called level anti-crossing point (LAC)$^{35-37}$, whose effect can be expressed as $|S^{P,P^*}, T^{I,I^*}\rangle \leftrightarrow |T^{P,P^*}, S^{I,I^*}\rangle$ occurs at the crossing point $(\nu_P - \nu_I) = J_{P,P^*} - J_{I,I^*} \pm 0.5 (J_{P,S} + J_{P,S^*} - J_{I,S} - J_{I,S^*})$. The states $|S^{I,I^*}\rangle$, $|S^{P,P^*}\rangle$ are the singlet states and $|T^{I,I^*}\rangle$, $|T^{P,P^*}\rangle$ denote the triplet states of the $I - I^*$ and $I_P - I_{P^*}$ pairs. The constants $J_{P,P^*}$, $J_{I,I^*}$, $J_{P,S}$, $J_{P,S^*}$, $J_{I,S}$ and $J_{I,S^*}$ represent the $J$-coupling constants between pairs $I_P - I_{P^*}$, $I - I^*$, $I_P - S$, $I_P - S^*$, $I - S$ and $I - S^*$. After the LAC process an intermolecular singlet state $|S^{I,I^*}\rangle$ is formed between the two molecules $I - S$ and $I^* - S^*$. In a further step the symmetry of $|S^{I,I^*}\rangle$ is broken by a CH activation mechanism in the N-heterocyclic carbene Ir-complex, where the methyl group of the IMES ligand coordinates to the Ir-para-hydrogen binding sites$^{39,40}$. This breaks the chemical and magnetic equivalence between the $S-I$ and $S^* - I^*$ target molecules, leading to $J_{I,S} \neq J_{I^*,S^*}$, $J_{I,S^*} \neq J_{I^*,S}$ and to a small chemical shift difference $\delta\nu$ between spins $I$ and $I^*$. As a result the $I - I^*$ singlet-order is broken up into different forms of hetero- and homo-nuclear spin-order terms, which can be approximated by two complementary
The quantities \( is proportional to the coefficients represented by the square brackets in equation (8).

\[
\rho_{I,1^S} = I_z I_z^* + \frac{1}{2} \left[ \frac{1}{1 + (y + x)^2} + \frac{1}{1 + (y - x)^2} \right] ZQ_x
\]

\[
+ \frac{1}{2} \left[ \frac{y + x}{1 + (y + x)^2} + \frac{y - x}{1 + (y - x)^2} \right] (I_z - I_z^*) \cdot S_z
\]

\[
+ \frac{1}{4} \left[ \frac{1}{1 + (y + x)^2} - \frac{1}{1 + (y - x)^2} \right] ZQ_x \cdot S_z
\]

\[
+ \frac{1}{4} \left[ \frac{y + x}{1 + (y + x)^2} - \frac{y - x}{1 + (y - x)^2} \right] (I_z - I_z^*)
\]

The lower dot between two operators, for example \( I_z, S_z \), denotes the direct operator product between \( I_z \) and \( S_z \). The first term \( I_z I_z^* \) in equation (8) represents homo-nuclear \( I-I^* \) two-spin order. \( ZQ_x = I_x, I_x^* + I_y, I_y^* \) denotes homo-nuclear zero-quantum spin-order. The terms \( I_z S_z \) and \( I_z^* S_z \) represent hetero-nuclear two-spin order between spins \( I-S \) and \( I^*-S \). The term \( ZQ_x, S_z \) corresponds to three-spin order and \( I_z - I_z^* \) represents homo-nuclear anti-Zeeman order. The complementary density matrix \( \rho_{I^*,1S} \) is introduced here since the IMES ligand on the Ir-complex is symmetric and its methyl group coordinates to other possible para-hydrogen binding sides of Ir. Therefore \( \rho_{I^*,1S} \) is obtained from equation (8) by replacing \( I_z \) by \( I_z^* \) and \( S_z \) by \( S_z^* \). Each spin-order has an amplitude which is proportional to the coefficients represented by the square brackets in equation (8).

The quantities \( y \) and \( x \) are defined as \( y = (J_{1S} - J_{1I^*} + J_{1^S} - J_{1^I}) / (2J_{1I^*}) \) and \( x = \delta \nu / J_{1I^*} \), where \( J_{1S}, J_{1I^*}, J_{1^S} \) and \( J_{1I^*} \) are the hetero-nuclear \( I-S, I^*-S^*, I^*-S \) and \( I-S^* \) \( J \)-coupling constants and \( J_{1I^*} \) is the homo-nuclear coupling constant between \( I \) and \( I^* \). At low field \((< 1 \text{ mT}) \) \( \delta \nu \) is small \((< 0.04 \text{ Hz}) \) compared to \( J_{1I^*} \approx 0.2 \text{ Hz} - 1 \text{ Hz} \) so \( x \ll y \). Thus the last two terms in equation (8) \( (ZQ_x, S_z \text{ and } I_z - I_z^*) \) can be neglected.

Similar arguments apply for \( \rho_{I^*,1S} \), especially the operator part in the third term of equation (8) becomes \( (I_z^* - I_z) \cdot S_z^* \). In summary in the bound complex only the first three terms \( I_z, I_z^*, ZQ_x \) and \( (I_z - I_z^*) \cdot S_z \) and their complementary terms in \( \rho_{I^*,1S} \) are different from zero, but they are NMR and raser silent. After dissociation of the Ir-complex all the correlations between spins \( I-I^*, S-S^*, I-S^* \) and \( I^*-S \) are lost, which means that only the two-spin order terms \( I_z S_z \) and \( I_z^* S_z \) survive. At this point it is important to realize that the reduction of the six spin system \( I_P, I_P^*, S, S^*, I, I^* \) which undergoes a symmetry break, into two three-spin systems \( \rho_{I^*,1S} \) and \( \rho_{I^*,1S} \) (see equation (8)) is a simplification which qualitatively describes the dominant spin order terms which are produced in the bound complex. Evaluation of the simpler four-spin system \( I, I^*, S, S^* \) leads to complex analytical expressions with less insight and with additional spin-order terms up to \( 4^{th} \) order. Nevertheless during symmetry break in \( I, I^*, S, S^* \), meaning \( J_{1S} \neq J_{1^I} \).
and $J_{1,S'} \neq J_{1,S}$, the main effect on the singlet $I - I'$ seen from the I-site is equivalent to a single spin K forming a three spin system $I, I', K$ where $J_{1,K} \neq J_{1,S'}, J_{1,K} = J_{1,S} - J_{1,S'}$ and $J_{1,K} = J_{1,S'} - J_{1,S}$ holds. The same arguments apply to the $I'$-site in the $I', I, K'$ three-spin system. This leads to the simplified model represented by equation (8). Expanding this model to the case of $^{15}$N-acetonitrile with three magnetically equivalent protons coupled to $^{15}$N leads to the following ansatz for the density matrix $\rho_{ac}(0)$ after dissociation:

$$\rho_{ac}(0) = c_2 S_z. (I_{2z} + I_{3z} + I_{4z}) + e_4 S_z. I_{2z}. I_{3z}. I_{4z} .$$  \hspace{1cm} (9)$$

$\rho_{ac}$ is composed of a sum of three equivalent two-spin order terms $S_z. I_{3z}, j = 2..4$ and a four spin order term $S_z. I_{2z}. I_{3z}. I_{4z}$ which might be generated in the full quantum mechanical treatment of the $(I_3S, I^*_3S^*)$-system. The coefficients $c_2$ and $e_4$ describe the magnitude of the corresponding spin order. For $^{15}$N-pyridine at $B_0 \approx 1$ mT we have $x \ll y_o, y_m, y_p$, where $o$, $m$ and $p$ stand for ortho, meta and para protons and $y_o, y_m, y_p$, might have different values (three different but overlapping level crossing terms). Therefore the density matrix can be expressed as

$$\rho_{py}(0) = \gamma_{2,o} S_z. (I_{2z} + I_{6z}) + \gamma_{2,m} S_z. (I_{3z} + I_{5z}) + \gamma_{2,p} S_z. I_{4z} + \text{higher spin order terms} .$$  \hspace{1cm} (10)$$

The coefficients $\gamma_{2,o}$, $\gamma_{2,m}$ and $\gamma_{2,p}$ describe the magnitude of the relevant $^1H-^{15}N$ two-spin order terms. Higher spin-order terms (four and six-spin order) are not given explicitly, since they have not been observed experimentally. A simulation shows that on the I-channel the four and six-spin order terms are not observable, but on the S-channel they could be detected, as shown later for the case of $^{15}$N-acetonitrile (see Fig. S3).

After 90°-pulse excitation on $^1$H ($^{15}$N) all $I_{jz}$ ($S_z$) operators in equations (9,10) become $I_{jx}$ ($S_x$). We denote the resulting density matrices after 90° pulse excitation as $\rho_k(\pi/2)$ where $k = ac$ or $py$. Evolution of $\rho_k(\pi/2)$ under the influence of the Hamiltonian given by equations (6,7) and introducing a damping term $\sim \exp(-t/T_2)$ in order to obtain finite line-widths leads to:

$$\rho_k(t) = \exp(-iH_k t) \rho_k(\pi/2) \exp(iH_k t) \exp(-t/T_2) .$$  \hspace{1cm} (11)$$

The observed $^1$H ($^{15}$N) signal decay (FID) can be calculated according to

$$\langle I_{k}^{\text{det}} \rangle = \text{Tr} \{ \rho_k(t) I_{k}^{\text{det}} \} .$$  \hspace{1cm} (12)$$

where $I_{k}^{\text{det}}$ is the $^{15}$N ($^1$H) detection operator $S_x (\sum I_{jx})$. The theory represented by equations (6–12) is the base for computer simulations where the evolution of the FID
is calculated successively by dividing the time $t$ into small enough discrete steps. The simulated FID can be Fourier transformed to obtain the NMR spectrum.

The model described above leads to the conclusion that for free $^{15}$N-acetonitrile and $^{15}$N-pyridine molecules the NMR spectrum is dominated by observable two-spin order terms while for bound molecules the $^1$H and $^{15}$N dipolar signal contributions are practically zero. This is because of the terms $(I_x - I_x^*).S_z$ and $I_x. I_x^*$ in equation (8), which have zero NMR signal contributions at low field. SABRE experiments with $^{15}$N labelled pyridine at earth’s field and subsequent observation in high-field confirm a $^1$H--$^{15}$N two-spin order on the $^{15}$N NMR signal of the free pyridine and no $^{15}$N-NMR signal for the bound pyridine species\textsuperscript{35,36}.

Finally at ultra-low fields ($B_0 < 1$ µT) another efficient pathway for polarization transfer exists from para-hydrogen to the $^{15}$N spins (SABRE-SHEATH\textsuperscript{41,42}), which leads to single-spin order of $^{15}$N with a high degree of polarization $P_{^{15}N} \approx 0.1$. SABRE-SHEATH can be explained as a LAC process between three spins $I_P$, $I_{P^*}$ and $S$, where the crossing condition at ultra-low field is given by $\nu_I - \nu_S = J_{P,P^*}$.

To prove whether the model described above, especially the two- and four-spin order given by equations (9,10), is an adequate description for the spin-physics involved in low field, several SABRE and raser experiments with $^{15}$N labelled acetonitrile and pyridine have been performed at 1 mT field.

Fig. S3a shows the measured $^1$H FID from SABRE polarized $^{15}$N-acetonitrile after 90° pulse excitation measured at $B_0 = 0.977$ mT ($\nu_H = 41.6$ kHz). Before the pulse the protons of $^{15}$N-acetonitrile are in a state just slightly above the raser threshold, as can be seen by the small oscillations in Fig. S3a and by the corresponding Fourier spectrum showing a single line (Fig. S3b, top). The FID after the pulse starts at zero amplitude and has a long lasting beat pattern, and the associated transverse decay time ($\approx 5$ s) is longer compared to the measured $T_2$ relaxation time ($\approx 0.7$ s). This can be explained by the anti-phase character of the two co-rotating magnetization components, which reduce their dipolar interaction with respect to each other. The spectrum of this long living FID (Fig. S3b, center) shows two lines with equal magnitudes but opposite sign, which are separated by $J_{^{15}N-H} = 1.7$ Hz. This measured antisymmetric $^1$H spectrum is in agreement with the simulation (Fig. S3b, bottom), in which pure two spin order $\approx S_z.(I_x + I_3x + I_4x)$ with a corresponding antisymmetric spectrum is assumed. We could also measure the corresponding $^{15}$N FID of $^{15}$N-acetonitrile at $B_0 = 9.65$ mT ($\nu_{^{15}N} = 41.6$ kHz) after 90° pulse excitation in a single scan (Fig. S3c), indicating that the SABRE process at low field also polarizes the $^{15}$N nuclei significantly. The FID which starts at zero and the observed beat pattern cannot be explained by pure single- or two-
Spin order. The corresponding Fourier transformed spectrum (Fig. S3 d, top) shows an antisymmetric quartet with line intensity ratio of 1 : 2 : (−2) : (−1). The corresponding simulated $^{15}$N spectrum (Fig. S3 d, bottom) is in agreement with the measured spectrum if $\rho_{ac}(0)$ from equation (9) is assumed with $c_2 = 1 \cdot 10^{-4}$ and $e_4 = -2 \cdot 10^{-4}$. For $^{15}$N-acetonitrile the $^1$H and the $^{15}$N-spectra at low field suggest that besides two-spin order there is a significant amount of four-spin order (both result in an anti-symmetric spectrum) and other single- and three-spin order terms (corresponding to a symmetric spectrum) are completely absent.

To compare the theory above with the measured $^1$H SABRE spectrum of $^{15}$N-pyridine and for the purpose of spectral assignment of the observed NMR lines to possible raser active modes with their associated spin states, different simulations were performed. Fig. S4a-j shows the simulated results for different initial conditions and spin configurations. The measured anti-symmetric $^1$H SABRE spectrum (grey) is plotted in each panel in order to compare with the simulated amplitudes and line positions. In Fig. S4 a-e (blue) pure two-spin order $I_{2z}.S_z$ between ortho $^1$H pyridine ($I_2$-spin) and the $^{15}$N-spin ($S$-spin) is assumed. The measured $I$-spin (here $I_2$) and some possible spin states of the other spins ($S$, $I_3 - I_6$) are indicated in the insets. The resulting line position for one specific spin state is marked by an arrow. In Fig. S4a the initial state is $I_{2z}.S_z$. After $90^\circ$ $^1$H pulse excitation the two-spin state is $I_{2x}.S_z$ and the corresponding simulated spectrum is an antisymmetric doublet separated by $J_{12} = -9.8$ Hz. In Fig. S4b one additional meta-proton $I_3$ is included, resulting in a strongly coupled three spin $S - I_2 - I_3$ system, and the corresponding simulated $I_2$-spectrum shows an additional splitting into two doublets (each splitted by $J_{23} = 5.05$ Hz). Due to strong coupling effects four additional lines (inter-combination lines) are visible close to the center frequency (41666 Hz). The result of the simulations for the strongly coupled four ($S - I_2 - I_3 - I_4$), five ($S - I_2 - I_3 - I_4 - I_5$) and six-spin ($S - I_2 - I_3 - I_4 - I_5 - I_6$) system including the para ($I_4$), meta ($I_5$) and ortho ($I_6$) spins are shown in Figs. S4c-e. One specific line position in Fig. S4e, where spin $I_3$ = up, $I_4$ = up, $I_5$ = down and $I_6$ = up (see inset), is exactly where the first raser line (41663.06 Hz) is observed (Fig. 3c, line 1). The third raser line (41663.32 Hz) results from a flip of spin $I_6$ to the down position, the expected frequency shift of $J_{26} = -0.26$ Hz being equal to the observed frequency difference between raser line 1 and 3. Fig. S4f shows the simulated spectrum (orange) for the case that both ortho-protons are in a two spin-state $\rho(0) = (I_{2x} + I_{6z}) . S_z$ and spin $I_2$, $I_6$ are detected after a $90^\circ$ pulse excitation resulting in $\rho(\pi/2) = (I_{2x} + I_{6z}) . S_z$. The simulated spectrum reflects basically all line positions of the measured SABRE spectrum but not the line intensities.
**Supplementary Information**

**Fig. S4:** Comparison between measured and simulated $^{15}$N-pyridine $^1$H spectra.

$^1$H (I-) spectrum measured at 41.6 kHz from $^{15}$N-pyridine (grey) and simulated $^1$H spectra (colour) based on a linear combinations of pure $^1$H–$^{15}$N two-spin order. Insets show initial density matrix $\rho(0)$ (upper left) and spin states including measured $I_j$-spins ($j = 2..6$, see the chemical structure insets). Simulation parameters: $(J^*_{12}, J^*_{13}, J^*_{14}) = (-9.8, -1.56, 0.18) \text{ Hz}, (J^*_{23}, J^*_{34}, J^*_{24}, J^*_{25}, J^*_{35}, J^*_{26}) = (5.05, 7.83, 1.78, 0.95, 1.38, -0.26) \text{ Hz}$, center frequency $\nu_I = 41666 \text{ Hz}, T_2 = 1.5 \text{ s}$. Arrows in (a–d) indicate the frequency for a specific spin configuration shown in the inset. Arrows in (e–g) correspond to the frequencies for the four raser transitions observed.
The largest deviations in line intensities appear close to the center frequency at 41 666 Hz and at the wings (41 657 Hz and 41 675 Hz). This indicates that the total initial state is not purely $\rho(0) = (I_{2z} + I_{6z}) \cdot S_z$ but other two- or higher-spin order contribution from the meta- and para- hydrogens have to be taken into account. Another specific line position associated with the spin state $I_3 = \uparrow$, $I_4 = \downarrow$, $I_5 = \uparrow$, as indicated in Fig. S4f, corresponds exactly to the second raser active line (line 2 in Fig. 3c) at 41 662.32 Hz. The simulated spectrum in Fig. S4g results from both meta-protons being in a two spin-state $\rho(0) = (I_{3z} + I_{5z}) \cdot S_z$ and spin $I_3 = \uparrow$ and $I_5 = \uparrow$ are detected (green). The fourth raser active line (line 4 in Fig. 3c) at 41 663.91 Hz with large negative amplitude and with $S = \uparrow$, $I_2 = I_6 = \downarrow$ and $I_4 = \uparrow$ can be identified. Fig. S4i is the result with the para-proton being in a two spin-state $\rho(0) = I_{4z} \cdot S_z$ while spin $I_4$ is detected (pink).

In summary the analysis of the position, amplitudes and sign of the lines at the region of raser activity ($\approx 41 662 \text{ Hz} - 41 664 \text{ Hz}$) in Figs. S4f,g,h lead to the conclusion that in these experiments only ortho- and meta-protons are raser active in $^{15}$N-pyridine. In addition all four raser active lines in Fig. 3c are identified with respect to their frequency position and their corresponding spin state. Figure S4i shows the complete simulated spectrum (red) assuming a superposition of ortho-, meta- and para-protons in a two spin state (see equation (10)). The state $\rho(0) = \gamma_{2,o} S_z \cdot (I_{2z} + I_{6z}) + \gamma_{2,m} S_z \cdot (I_{3z} + I_{5z}) + \gamma_{2,p} S_z \cdot I_{4z}$ with $(\gamma_{2,o}, \gamma_{2,m}, \gamma_{2,p}) = 10^{-4} \cdot (1, 0.5, 0.5)$ fits best to the measured SABRE spectrum. The small deviations can be explained by the assumption that in the simulations all linewidths are assumed to be equal ($T_2 = 1.5 \text{ s}$). This is not the case in the measured SABRE spectrum, where different lines have different widths. Especially two specific raser lines (at 41 663 Hz and 41 663.26 Hz) show large narrow peaks, while the mirrored lines on the other side of the spectrum at 41 668.74 Hz and 41 669 Hz are broader.

Finally Fig. S4j shows the simulated spectrum (brown) if all protons of the $^{15}$N-pyridine are prepared in a single-spin order $\rho(0) = o (I_{2z} + I_{6z}) + m (I_{3z} + I_{5z}) + pI_{4z}$ with $o = -0.07$, $m = -0.04$ and $p = -0.08$. The spectrum is symmetric and not compatible with any observed SABRE spectrum.

VI Multi-mode EHQE based raser

While a simulation of the amplitudes of the SABRE NMR lines is possible, a model to describe the four observed raser amplitudes at different flow rates is not possible yet. This is due to the complexity of the multi-mode raser operation which in our case involves up to four non-linear coupled differential equations. A qualitative explanation for the increasing number of raser modes with increasing $p$-H$_2$ flow rate is possible (Fig. 3b,c). Each mode has its own raser threshold that is crossed in sequence as their polarization increases with rising flow rate. At higher flow rate there is a better mixing of $p$-H$_2$ gas
bubbles with the methanol solution, resulting in a higher \( p\)-\( H_2 \) concentration at the site of the dissolved catalyst. This increases the degree of polarization on the target molecule. It is expected that at very high degrees of polarizations using an improved experimental setup all raser modes can be observed. This means that the whole NMR SABRE spectrum can be recovered via its antisymmetric properties with a vastly improved frequency resolution from the raser modes. From the difference in frequency between the four lines \( \Delta \nu_{i,j} \) with \( i,j = 1..4 \) homonuclear \( J \)-coupling values as well as linear combinations of \( J \)-coupling constants can be deduced. The most narrow line shown in Fig. 4b, corresponds to the two mode raser and the frequency difference is \( \Delta \nu_{1,2} = J_{24}^* - J_{25}^* + J_{26}^*/2 = 0.696 \) 80 Hz measured over 1000 s. In Fig. 4c, the frequency differences of the four mode raser are shown. Especially \( \Delta \nu_{1,3} = J_{26}^* = -0.2561 \) Hz corresponds to a spin flip of \( I_6 \) while \( I_2 \) is measured (see spin states 1 and 3 in Fig. 3c). Note the frequency difference \( \Delta \nu_{1,2} = J_{24}^* - J_{25}^* + J_{26}^*/2 = 0.7357 \) Hz from Fig. 4c differs slightly from Fig. 4b because of slightly different experimental conditions. A spin flip of \( I_6 \) leads to \( \Delta \nu_{2,3} = J_{24}^* - J_{25}^* - J_{26}^*/2 = 0.9918 \) Hz as shown in Fig. 3c and Fig. S4 e. The frequency difference \( \Delta \nu_{2,4} \) in Fig. 4c cannot be interpreted as combination of \( J \)-couplings since it arises from the difference between the ortho- and meta-raser modes we have identified in Fig. S4 f,g.

We remark that the values of \( J \)-coupling constants \( J_{ik}^* \) for the simulations are very close (up to 0.2 Hz) to the \( J \)-coupling constants of \( ^{15}\)N-pyridine measured in high field\(^{20}\). The variation can be explained by different experimental conditions, such as the temperature, pyridine and catalyst concentration. The experimenters are aware of the effect of ‘frequency pulling’ in masers and seek to quantify it in future raser experiments.

Another interesting property of the multi-mode raser is the near-chaotic appearance of its waveform recorded over time. Actually, the signal shown in Fig. 3b bottom (orange) line looks quite chaotic but can be synthesized by a superposition of four sinusoids, each with their own frequency and amplitude. We used Python and MatPlotLib\(^{44,45}\) in the interactive program \texttt{SimNatPhysMain3.py} which is available in another SI file to produce the data given in Fig. S5. For optimum comparability with the experiment, the four frequencies and amplitudes can be extracted from the spectrum shown in Fig. 3c. We took advantage of the precisely tabulated values coming straight from the Fourier transform to determine the eight parameters with exceedingly high precision. Fig. S5a shows a plot of 100 seconds of the sum of the four sinusoids. Fig. S5b shows a time slice from 15 to 40 seconds of the simulated data, the peak patterns look distinctively repeated yet are each unlike each other. On the scale 18.74 s–28.74 s (Fig. S5c) the signal looks very much like the experimental recording (Fig. S5d). In summary, the superposed sinusoids completely describe the time evolution of the raser signal. Non-linearities\(^{30}\) are not involved in explaining the waveform observed. At very high polarizations and \( Q \)-factors, real chaotic behaviour may emerge.
Fig. S5: Simulated and measured $^{15}$N-pyridine $^1$H four mode raser signal.

**a**, Raser signal simulated as a linear superposition of sinusoids plotted over 100 s. $S(t) = A_1 \cdot \sin(2\pi \cdot t \cdot 31.08864 \text{ Hz}) + A_2 \cdot \sin(2\pi \cdot t \cdot 31.821968 \text{ Hz}) + A_3 \cdot \sin(2\pi \cdot t \cdot 32.077521 \text{ Hz}) + A_4 \cdot \sin(2\pi \cdot t \cdot 32.599739 \text{ Hz})$ with $A_1 = 0.381134327, A_2 = 0.306345952, A_3 = 0.180330163, A_4 = 0.132189557$.  

**b**, Enlarged section from **a** (15s–40s) showing repeating patterns that are modulated by very low frequencies. 

**c**, Section from **b** (18.74s–28.74s) showing a signal comparable to the experimental result in **d** which is the same as main Fig. 3b, bottom line (orange).
The fundamental limit of precision for frequency measurements in a raser follows from the Cramér-Rao lower bound (CRLB)\(^43\) for the standard deviation of frequency \(\sigma_f\):

\[
\sigma_f \geq \frac{\sqrt{3}}{\pi SNR \sqrt{f_{BW} T_m^3}} ,
\]

(13)

where \(SNR\) denotes the signal to noise ratio, \(f_{BW}\) is the detection bandwidth and \(T_m\) is the measurement time. Equation (13) can be used as an estimate for the lower bound \(\sigma_f^{\text{raser}}\) of the \(^{15}\text{N}\)-pyridine raser at 1 mT field (Fig. S5d, Fig. S6a). Given a bandwidth \(f_{BW} = 100\text{ Hz}\) and \(T_m = 1000\text{ s}\), \(SNR = 50\) (Fig. S6a) or \(T_m = 100\text{ s}\), \(SNR = 25\) (Fig. S5d), we obtain for the lower bound \(\sigma_f^{\text{raser}} = 3.6 \cdot 10^{-8}\) Hz (Fig. S6a) and \(\sigma_f^{\text{raser}} = 2.2 \cdot 10^{-6}\) Hz (Fig. S5d). Following equation (13) these two limits can theoretically be further lowered by either increasing the measurement time \(T_m\) or the \(SNR\). Let us compare two lower bounds \(\sigma_f^{\text{raser}}\) for the raser with \(\sigma_f^{\text{NMR}}\) for a corresponding NMR experiment with \(N = T_m/\Delta t_m\) single scans, where each scan has a duration \(\Delta t_m\). According to [25] \(\sigma_f^{\text{NMR}}\) is given by replacing \(T_m\) by the transverse relaxation time \(T_2\) in equation (13). The improvement of the \(SNR\) for \(N\) scans is \(SNR \cdot \sqrt{N} = SNR \cdot \sqrt{T_m/\Delta t_m}\). If we assume equal \(SNR\) and bandwidth \(f_{BW}\) for the NMR signal per scan and the raser signal, the ratio is given by:

\[
\frac{\sigma_f^{\text{raser}}}{\sigma_f^{\text{NMR}}} = \frac{T_2^{3/2}}{T_m \sqrt{\Delta t_m}} .
\]

(14)

For the case of the \(^{15}\text{N}\)-pyridine raser \(T_2 = 1\text{ s}\), \(\Delta t_m = 10\text{ s}\) (inverse of SABRE pumping rate 1/\(\tau_p\)) and \(T_m = 1000\text{ s}\) the ratio becomes \(\sigma_f^{\text{raser}}/\sigma_f^{\text{NMR}} \approx 1/(3.3T_m) \approx 3.3 \cdot 10^{-4}\). Therefore for large enough \(T_m\) raser-operation has a frequency precision which is orders of magnitude higher compared to the NMR experiment with \(N = T_m/\Delta t_m\) scans.

The precision of our measurements is far away from the bounds given by equation (13). In order to get an idea about the statistical fluctuations and the long term behaviour of a specific \(J\)-coupling constant of \(^{15}\text{N}\)-pyridine (two mode raser), a \(^1\text{H}\) raser signal is measured over a time of 1200 s. The data measured at a constant \(p\)-\(\text{H}_2\) flow rate of 100 cm\(^3\) min\(^{-1}\) is shown in Fig. S6a. A beat pattern consisting of two lines with similar amplitude is observed over the whole measurement time (Fig S6a, inset). The Fourier transformed spectrum of the entire raser signal shows two groups of broadened lines with nearly identical shape and amplitude (Fig. S6b). The broadening is caused by the magnetic field fluctuations over a time scale of 1200 s. The observed splitting of \(\approx 0.7\) Hz is caused by simultaneous oscillation of two raser modes with different frequencies.
Fig. S6: $^1$H raser signal of $^{15}$N-pyridine $^1$H measured at 41.6 kHz and for 1200 s. 

a, The signal amplitude slowly decreases due to evaporation of liquid methanol. The $p$-H$_2$ flow rate is 100 cm$^3$ min$^{-1}$. Inset is a section of 30 s duration showing two modes oscillating simultaneously. b, Fourier transformed spectrum of entire raser signal in a, resulting in two broad lines caused by magnetic field fluctuations. c, Evolution of line splitting $\Delta \nu_{1,2}$ from 0.7004 Hz to 0.6858 Hz free of magnetic field noise versus time. The data was obtained by dividing the $^1$H raser signal in a into time blocks of 30 s and by applying a Fourier transformation and a correction of magnetic field fluctuations. The error bar of each data point is $\approx 2 \cdot 10^{-4}$ Hz. The red line is for guiding the eyes.
Based on the observed frequencies and the experimental conditions we conclude that these two modes correspond to the lines 1 and 2 in Fig. 3c. From the associated spin-states of \(^{15}\)N-pyridine (Fig. 3c, insets 1,2) the frequency difference of modes 1 and 2 is given by \(\Delta \nu_{1,2} = J_{24}^* - J_{25}^* + J_{26}^*/2\).

Fig. S6c shows the evolution of \(\Delta \nu_{1,2}\) versus time. The data was obtained by dividing the \(^1\)H raser signal in Fig. S6a into time blocks of 30 s, applying the correction of magnetic field fluctuations and measuring the frequency in the Fourier transformed spectrum. As the main trend \(\Delta \nu_{1,2}\) decreases from 0.7004 Hz at \(t = 0\) to 0.6858 Hz at \(t = 1200\) s.

Sudden fluctuations of \(\pm 2\) mHz on the 100 s time scale are superimposed on the long term trend. We believe that the slow decrease is due to constant evaporation of the solvent (\(d_4\)-methanol) which leads to a continuous increase of the \(^{15}\)N-pyridine and Ir catalyst concentration. The dependence of \(J\)-coupling constants on the \(^{15}\)N-pyridine concentration is known from literature\(^{20}\), but has never been reported for highly diluted solutions in the milli-Hz regime. The data in Fig. S6c implies that the linewidths \(\Delta \nu\) of the measured raser signals from Fig. 4b,c (2.6 mHz and 13 mHz, respectively) for a given time interval \(\Delta t\) (800 s and 130 s, respectively) is determined by systematic fluctuations in concentration of the raser active molecules in solution and not by the inverse of the measuring time \(\Delta \nu = 1/(\pi \Delta t)\). In future a significant improvement in precision which comes closer to the bound given by equation (13) is expected by using a better continuous flow apparatus eliminating the concentration dependence effect.

VIII Temperature dependence of raser and SABRE signals

A major obstacle for all technology involving para-hydrogen is the need for cryogenic temperatures (\(T < 123\) K) to produce enriched para-hydrogen from the 3:1 ortho-/para-hydrogen mixture at room temperature\(^{15}\). The cryo-coolers (e.g. Gifford-McMahon cycle cooler or pulse tube) working at \(T \approx 30\) K require a lot of input power and thick insulation, resulting in noisy and bulky equipment. This is a serious disadvantage for realizing a miniaturized SABRE based raser, even though the sample is kept at room temperature. Therefore starting from 60 K we tested the limit up to which conversion temperature (diminishing \(p\)-H\(_2\) concentration) SABRE signals and associated raser operation are observable in a continuous mode (Fig. S7). We found raser activity for pyridine up to 155 K. The \(SNR \approx 8000\) for the \(^1\)H SABRE signal at 60 K corresponds to a proton polarization \(P_H \approx 1 \cdot 10^{-2}\) and \(^1\)H signal is observed up to 270 K. Considering the linewidth \(\Delta\) of the SABRE spectra versus the \(p\)-H\(_2\) conversion temperature (blue triangles in Fig. S7), we observe a knee at \(T \approx 155\) K. Above this temperature, the linewidth \(\Delta \approx 0.4\) Hz is constant. Below \(T = 155\) K, \(\Delta\) rises exponentially with falling conversion temperature. The observed linewidth is given by \(\Delta \approx \frac{1}{\pi} \left( \frac{1}{\tau_s} + \frac{1}{\tau_{rd}} \right)\) where \(\frac{1}{\tau_{rd}} = \frac{\mu_s \hbar}{4 \eta Q} \gamma_H^2 n_s |P_H|\) is...
proportional to the proton polarization $P_H$ which is proportional to the molar fraction of $x_p$ of $p$-H$_2$ in the feed gas flow. The molar fraction $x_p$ is well approximated by an exponential function in the temperature range from 50 K–150 K, e.g. $x_p \propto \exp(-E_a/kT)$ where $E_a$ is the activation energy for conversion from ortho- to para-H$_2$. Therefore, the rising part of $\Delta$ below 155 K reflects the ortho-para activation energy of hydrogen. The position of the knee (our limit for raser activity) at $T = 155$ K could be raised using better EHQE circuits and para hydrogen delivery to a temperature of $T > 200$ K, where small Peltier coolers can be applied.

Fig. S7: $^1$H SNR from 1 $\mu$L pyridine measured at 166 kHz versus conversion temperature inside p-H$_2$ generator for SABRE and raser signals. SNR of steady state $^1$H raser signal (circles) and of corresponding SABRE FID after 30 s crusher pulse and subsequent 90° pulse excitation (squares). The $p$-H$_2$ flow is 35 cm$^3$ min$^{-1}$ at 4 bar. The linewidth $\Delta$ is determined from the Fourier-transformed spectrum of the SABRE FID (triangles). Solid lines are for guiding the eyes. With decreasing temperature raser activity starts at 155 K. At this point the radiation damping rate is equal to the $^1$H linewidth of the SABRE FID (equation (1)). At even lower temperatures the $^1$H linewidth increases exponentially, reflecting the energy of activation between ortho- and para-hydrogen states.
References