Emergent Dirac fermions and broken symmetries in confined and deconfined phases of $\mathbb{Z}_2$ gauge theories

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I. LATTICE MODEL

For clarity, in Fig. S1, we provide a graphical illustration of the lattice model presented in the main text (Eqs. 1 and 2) and in Fig. S2 we depict the physical on-site configurations satisfying the Gauss law constraint in the “even” sector (see Eq. (3)) with $Q = 1$.

II. PHYSICAL DEGREES OF FREEDOM

To relate our work to more familiar condensed matter models, in this section we discuss the gauge invariant physical degrees of freedom of our model. In doing so, it is important to distinguish between Ising gauged charged and neutral operators.

We first consider gauged operators, see Table. I. Here, similarly to the pure IGLT, we identify two bosonic degrees of freedom, the ‘m’ and ‘e’ particles and a fermion. The ‘m’ particle (“vison” $S^1$) corresponds to an Ising $\pi$-flux excitation that is created by a non local string operator. The fermion is identified with the operator, $c_\alpha$, which unlike the pure IGLT also carries a $U(1)$ charge and a $SU(2)$ spin. This important difference, in the absence of pairing, allows us to define a Fermi surface and gives rise to the Deconfined Dirac phase. The ‘e’ particle is then a bound state of the vison and fermion.

We now discuss the gauge invariant operators, see Table.II, which are the physical degrees of freedom. This necessarily involves even powers of the fermion operator $c$, which can carry charge 2 eg. $c^\dagger c^\dagger$ or charge 0 eg. $c^\dagger c$ which in turn can either transform as spin 1 or spin 0.

In the Ising sector, gauge invariant operators correspond to energy density operators such the Ising magnetic flux or the Ising electric field.

The physical degrees of freedom in our model are readily identified deep in the confined phase when $h \to \infty$ in Eq. 1. There, we would like to set all the $\sigma^z = +1$. The “even” sector constraint, $Q = 1$, then implies that the fermion density on a site $n_f^i = 0$ or 2, i.e. one has empty sites and sites occupied by a gauge neutral boson. In other words this is just the Hilbert space of a hardcore lattice boson model.

On the other hand, if we had imposed the ‘odd’ constraint - i.e. $Q = -1$, then deep in the confined phase we would obtain $n_f^i = 1$, and this could correspond to either a spin up or down fermion, which implies that we are dealing with a spin model.

Our physical (gauge invariant) degrees of freedom then correspond to bosons with a global $U(1)$ charge, and neutral spin excitations with integer spin. Therefore we are dealing with a boson only lattice model with spin and charge degrees of freedom. Importantly there are no gauge invariant fermions - so this is not explicitly an electronic model.

This is in contrast to the ‘orthogonal metal’ to which our phase has many similarities (such as a Fermi surface of $\mathbb{Z}_2$ gauge charged fermions, carrying a global $U(1)$ charge and spin $1/2$ $S^2$) but differs in that the models of $S^2$ explicitly involve electrons in the physical Hilbert space. It would be an interesting exercise to reintroduce electrons as an additional degree of freedom to bring this closer to modeling correlated systems.

FIG. S1: Lattice model of Eqs. (1) and (2) : The Ising gauge fields $\sigma^z_b$ (red arrows) reside on the links of a square lattice. Frustrated plaquettes, with $\prod_{b \in \square} \sigma^z_b = -1$, are marked in green. The sign of the fermion hopping amplitude is determined by the Ising gauge field $\sigma^z_b$ along the bond $b$. 
III. DETERMINANT QUANTUM MONTE CARLO SIMULATION

In this section, we provide a detailed derivation of the DQMC path integral formalism that incorporates Gauss’s law, we present the “zero problem” at half filling and explain our numerical solution, we describe our global updating scheme and finally we benchmark the QMC results with exact diagonalization (ED).

A. Path integral formulation

In this section we derive an explicit mapping of the two dimensional quantum problem to a three dimensional classical statistical mechanics model. This is done by rewriting the grand canonical partition function \( Z(\beta, \mu) \) (Eq. (4)) in terms of an imaginary time path integral. The procedure closely follows the standard QMC methods with the exception that in our case we must also incorporate the constraint (Eq. (3)).

We use a Trotter decomposition to write the thermal density matrix as \( e^{-\beta H} = \prod_{\tau=0}^{M-1} e^{-\epsilon H} \) with \( \epsilon = \beta/M \) and introduce resolution of the identities in the \( \sigma^z \) basis, \( I = \sum \sigma^z |\sigma^z\rangle \langle \sigma^z| \), between each imaginary time step,

\[
Z(\beta, \mu) = \sum_{\sigma^z_{\tau=0}^{\tau=M-1}} Tr_{\tau} \left[ \prod_{\tau=0}^{\tau=M-1} e^{-\epsilon H} |\sigma^z_{\tau=M-1}\rangle \langle \sigma^z_{\tau=0}| \right].
\]

In the above equation we use a unified space time notation, such that \( r = \{x, y, z\} \) and the temporal Ising gauge field is then,

\[
\sigma^z_{\tau=(x,y)} = \begin{cases} 
\lambda_r & \tau = M - 1 \\
1 & \text{else}
\end{cases}
\]

We note that at finite temperature, periodic boundary conditions along the imaginary time axis leads to a non trivial cycle. The temporal gauge field, therefore, can not be completely eliminated.

Following standard techniques, we can compute the matrix elements appearing in Eq. S1 to order \( \mathcal{O}(\epsilon^2) \). We focus on the first term containing the projection operator,

\[
\langle \sigma^z_{\tau=0}| \hat{P}_{\lambda_r} e^{-\epsilon H} |\sigma^z_{\tau=M-1}\rangle = e^{i\frac{\pi}{2} \lambda_r} e^{\epsilon H} W_{\sigma^z_{\tau=M-1}} + \mathcal{O}(\epsilon^2)
\]

The imaginary time depended fermion Hamiltonian is
given by,

\[ \mathcal{H}_f \left[ \sigma^z_{r-M-1} \right] = \sum_{r} K_{r,r^+\eta} \sigma^z_{r,\alpha} \sigma^z_{r^+\eta,\alpha} + h.c. \]

\( - \mu \sum_{r} \sigma^{\dagger}_{r,\alpha} \sigma_{r,\alpha} \) \hspace{1cm} (S4)

Explicitly, the kernel matrix equals \( K_{r,r^+\eta}(\tau = M - 1) = -t \sigma^z_{r,r^+\eta} \).

The Boltzmann weight associated with each gauge field configuration, \( W_{\sigma^z_{r-M-1}} = e^{S_{\sigma^z_{r-M-1}}} \), is given by the classical action,

\[ S_{\sigma^z_{r-M-1}} = \gamma \sum_{r \in \Box_{r,\tau,\eta}} \prod_{b \in \Box_{r,\tau,\eta-M-1,\eta}} \sigma^z_b + eJ \sum_{r \in \Box_{r,\tau,\eta}} \prod_{b \in \Box_{r,\tau,\eta-M-1,\eta}} \sigma^z_b. \] \hspace{1cm} (S5)

where \( \gamma = -\frac{1}{2} \log(tanh \epsilon h) \). In the first term, the plaquette \( \Box_{r,\tau,\eta} \) is a spatio-temporal plaquette defined by the space time point \( r = (r, \tau) \) and the direction \( \eta \). For instance, \( \Box_{r,\tau,\eta} \) corresponds to the set of bonds \( b = \{ (r, \tilde{e}_x), (r, \tilde{e}_y), (r+\tilde{e}_y, \tilde{e}_x), (r+\tilde{e}_x, \tilde{e}_y) \} \). In the second term, the plaquette is a planar plaquette defined similarly to Ising magnetic flux term of the Hamiltonian.

For the rest of the time slices the Boltzmann weight is readily evaluated in a similar manner. The temporal gauge field in this case is trivial \( \sigma^z_{r,\tau} = 1 \).

The fermionic weight amounts to tracing over a product of quadratic fermion propagators\(^{S3}\),

\[ w_f(\{ \sigma^z_{r,\eta} \}) = Tr_f \left[ e^{i \frac{T}{2} \sum_{\tau}(1-\lambda_\tau)\sigma^z_{r,\tau} \sum_{\tau} e^{-\epsilon H[\sigma^z_{r,\tau}]} } \right] \]

\[ = \det \left( I + P[\lambda_\tau] \sum_{\tau} e^{-\epsilon K(\tau)} \right) \] \hspace{1cm} (S6)

Here, the projector is manifested by the diagonal matrix \( P[\lambda_\tau] \) with elements \( P_{r,r} = \lambda_\tau \). For future convince we also define the equal time single particle Green’s function, which for a given gauge field configuration equals,

\[ G \left( I + P[\lambda_\tau] \sum_{\tau} e^{-\epsilon K(\tau)} \right)^{-1} \] \hspace{1cm} (S7)

The total weight of the fermionic sector is then a product over the spin up and down sector,

\[ W_f = w^2_f = \left[ \det \left( I + P[\lambda_\tau] \sum_{\tau} e^{-\epsilon K(\tau)} \right) \right]^2 \] \hspace{1cm} (S8)

Since both determinants are real, the weight in strictly non negative and hence free from the numerical sign problem.

**B. Particle-Hole symmetry and zero modes**

At zero chemical potential, \( \mu = 0 \), both the Hamiltonian \textit{and} the constraint are symmetric under the particle hole (PH) transformation \( C \), defined by,

\[ c_{\tau,\alpha} \rightarrow \begin{cases} c^\dagger_{\tau,\alpha} & \text{if } r \in A \\ -c_{\tau,\alpha} & \text{if } r \in B \end{cases} \] \hspace{1cm} (S9)

where the \( A \) and \( B \) sub-lattices correspond to the usual checkerboard division of the square lattice (or more generally any bipartite lattice) to two disconnected sub-lattices. PH symmetry has a dramatic effect on the fermionic configuration weight Eq. (S6). To see that, we apply the PH transportation, without loss of generality, only on the spin up, \( \alpha = \uparrow \), sector of the fermionic weight in Eq. (S6). We denote this operator by \( C_\alpha = \uparrow \). Since the Hamiltonian is symmetric under PH, the only non-trivial transformation is due to the constraint. Explicitly,

\[ C_{\uparrow} e^{i \frac{T}{2} \sum_{\tau}(1-\lambda_\tau)\sigma^z_{\tau,\uparrow} \sum_{\tau} e^{i \frac{T}{2} \sum_{\tau}(1-\lambda_\tau)\sigma^z_{\tau,\uparrow}} } \]

\[ C_{\uparrow} = \sum_{\tau}(1-\lambda_\tau)\sigma^z_{\tau,\uparrow} C_{\uparrow} = e^{i \frac{T}{2} \sum_{\tau}(1-\lambda_\tau)\sigma^z_{\tau,\uparrow} \sum_{\tau} e^{i \frac{T}{2} \sum_{\tau}(1-\lambda_\tau)\sigma^z_{\tau,\uparrow}} } \] \hspace{1cm} (S10)

As a direct consequence, if the parity of the temporal Ising gauge field \( P_\lambda = \prod_{\tau} \lambda_\tau \) is odd, \( P_\lambda = -1 \), the fermion weight obeys, \( W_f^\dagger(\{ \sigma_{r,\eta} \}/P_\lambda = -1) = -W_f^\dagger(\{ \sigma_{r,\eta} \}/P_\lambda = -1) \) and hence it must vanish.

The vanishing of the fermion determinant indicates on the presence of a \textit{finite} temperature fermionic zero mode. This result is surprising, since due to the anti-periodic boundary conditions along the imaginary time axis the lowest Matsubara frequency of fermions is non vanishing, \( T\pi \), and hence can not sustain poles on the real frequency axis.

We note that in our case the projection operator couples the temporal Ising gauge field to the density operator and acts as an effective complex chemical potential. In the odd sector, this effect shifts the lowest Matsubara frequency by \( T\pi \) down to zero and gives rise to a zero mode.

Naively, the above result does not affect the Monte Carlo sampling since it merely leads to a vanishing probability for configurations with odd parity. However, it gives rise to a systematic bias in computing expectation values of observables that are not symmetric under PH transformation of a single spin flavor, \( C_{\uparrow,\downarrow} \).

To address this problem, we introduce an extended configuration space which enables us to sample the contribution of the odd sector. For concreteness, we consider the pairing susceptibility. The derivation can be readily generalized to other observables.

The expectation value of the equal-time paring suscep-
tibility is given by,

$$P_{SC}(\mathbf{r}, \mathbf{r}, \tau = 0) = \frac{1}{Z} \sum_{\sigma_{\lambda,\eta}} W_{\sigma_{\lambda,\eta}}^{x} \text{Tr}_f \left[ e^{i \sum_{\tau} \mathcal{H}_{\tau}(\sigma_{\lambda,\eta})} \right] \times \prod_{\tau} e^{-\epsilon_{\tau}^{(\mathbf{r})}} \left| b_{\mathbf{r}}^{\dagger} b_{\mathbf{r}} \right|$$

(S11)

where $b_{\mathbf{r}}^{\dagger} = c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}}$. The above can be readily evaluated using Wick’s theorem.

As shown before, In the odd sector, the fermionic weight vanishes and hence the Green’s function $G$ diverges. The product, however, is finite. One possible solution for circumventing the ratio of zeros problem is to perform the MC simulation on a set of decreasing but finite chemical potentials. The zero chemical potential result by taking the limit $\mu > 0$, rendering our calculation significantly the usual DQMC algorithm since the SVD decomposition is available as part of the stabilization scheme of DQMC$^S$.

In the following we will introduce a simple solution that does not require breaking of PH symmetry. We first artificially break PH symmetry by introducing a small but finite chemical potential $\mu > 0$, rendering our calculation regular. In the last step we will recover the zero chemical potential result by taking the limit $\mu \to 0$ analytically. The odd sector contribution involve the finite product,

$$G^H = \det \left( G^{-1} \right) G$$

(S13)

In the above equation, we identified the product with the adjugate matrix$^S$.

To evaluate the product we must eliminate the singularity. This can be achieved by a singular value decomposition (SVD) analysis$^S$. Explicitly, we write $G^{-1} = U D V^T$ where $U, V$ are orthogonal matrices and $D$ is diagonal matrix with positive entries $D_{i,i} = d_i$ known as the singular values. We substitute the SVD decomposition in Eq. (S13) and obtain,

$$G^H = \det(U) \det(V) \det(D) V D^{-1} U^T$$

(S14)

In the odd sector, one of the singular values, $d_k$, vanishes in the limit of $\mu \to 0$. First we isolate the vanishing singular value $\det(D) = \left( \prod_{i \neq k} d_i \right) \times d_k$. Now we can cancel the singularity appearing in $D^{-1}$,

$$\lim_{\mu \to 0} (d_k D^{-1})_{\ell,\ell} = \lim_{\mu \to 0} \frac{d_k}{d_\ell} = \begin{cases} 1 & \ell = k \\ 0 & \text{else} \end{cases}$$

(S15)

Finally we obtain,

$$G^H = \det(U) \det(V) \left( \prod_{i \neq k} d_i \right) v_k u_k^T$$

(S16)

where $v_k(u_k)$ correspond to the $k$’th column of the matrix $V(U)$.

The above analysis suggests the following Monte Carlo sampling scheme. We consider an extended configuration space $\tilde{Z} = Z_{\text{even}} + Z_{\text{odd}}$. The configuration weight and Green’s function of the even sector, $\mathcal{P}_{\lambda} = 1$ are the same as the ones given in Eq. (S6) and Eq. (S7) respectively. For the odd sector, $\mathcal{P}_{\lambda} = -1$ we used Eq.(S16) to redefine both the configuration weight and the Green’s function for the odd sector. Explicitly,

$$w_f^{\text{odd}} = \prod_{i \neq k} d_i, \quad G^{\text{odd}} = v_k u_k^T$$

(S17)

Since we sample with respect to an extended configuration, $\tilde{Z}$, we must use reweighting to correctly compute expectation values. This is readily achieved by sampling the fraction of the even sector configurations $\langle \delta_{\text{even}} \tilde{Z} \rangle_{\tilde{Z}} = \frac{Z_{\text{even}}}{Z}$, such that,

$$\langle O \rangle_{\tilde{Z}} = \frac{\langle O \rangle_{\tilde{Z}}}{\langle \delta_{\text{even}} \tilde{Z} \rangle_{\tilde{Z}}}$$

(S18)

We note that the above scheme does not modify significantly the usual DQMC algorithm since the SVD decomposition is available as part of the stabilization scheme of DQMC$^S$.

C. Updating scheme

To evaluate the partition function in Eq.(4), we must devise an efficient scheme for sampling the configuration space $\{ \sigma_{\tau,\eta} \}$. To achieve that, we use both a local updating approach$^{S3}$ and a global updating strategy inspired by the worm algorithm (WA)$^{S6}$. The local updates, involve single spin flip of the Ising gauge fields. Both the temporal and spatial updates can be performed efficiently using a low rank (rank one in the case of the temporal link and rank two in the case of the spatial link) updating of the determinant in Eq. (S6) and the corresponding Green’s function.

Empirically, we found that using solely local updates does not lead to a sufficiently short MC correlation time. This effect is prominent near the critical point where the dynamics is critically slowed down due to the diverging correlation length. To tackle this problem, we introduce an additional MC move based on the highly efficient WA.

We reformulate the Ising gauge field sector of the action in a dual closed loop representation$^{S7}$. We note that this mapping is used in deriving the classical statistical mechanics duality between the classical Ising gauge theory and Ising model in three dimensions.

The closed loop configurations are constructed as follows. We first identify all frustrated space-time plaquettes $\square_{\tau}$ satisfying $\prod_{k \in \square_{\tau}} \sigma_{\tau}^{\square} = -1$. We then draw a line connecting the two neighboring sites of the dual three
dimensional cubic lattice that share the frustrated plaquettes, see Fig. S3. Since the ILGT is free of magnetic monopoles, the net flux through each elementary cube must be even. Therefore, the number of dual lattice lines emanating each dual lattice site (located at the center of the direct lattice cube) must be also even. This constraint enforces the lines to form a closed loop configuration\textsuperscript{\textnormal{35}}. Periodic boundary conditions along the spatial and temporal directions give rise to an additional constraint. The net flux along each plane must be even. This is in contrast to the closed loop representation of the classical Ising model, for which the total parity of the loops can fluctuate. A similar construction was proposed in the context of $U(1)$ gauge theory\textsuperscript{\textnormal{36}}.

The closed loop ensemble can be efficiently sampled using the WA, where the worm head flips the flux through the plaquette and generates arbitrary flux tubes. The loop fugacity is anisotropic and is determined according the Ising gauge field action (Eq. (S5)). During the loop update we ignore the fermionic weight $W_f$. After the loop is closed it is reintroduced in the acceptance probability of the entire move. In case that the move was accepted we translate the flux configuration to a gauge configuration. In the temporal gauge, this can be done uniquely up to a global gauge freedom in each space-time direction which is drawn randomly

D. Benchmarking against exact diagonalization

We verify the correctness of our numerical scheme by comparing the QMC results with exact diagonalization (ED) on a small system with $L = 2$. As concrete microscopic parameters we take $\beta = 2, t = 1, J = 1, \mu = 0$ and consider a set of eight evenly spaced points in $h \in [0.4, 1.0]$. The ED is performed by diagonalizing the Hamiltonian Eq. (1) restricted to the subspace of physical states satisfying the constraint in Eq. (3).

In Fig. S4 we compare the average kinetic energy, $\langle -K_x \rangle$ and the average Ising magnetic flux $\langle B \rangle$. We find excellent agreement within the statistical error. In Fig. S5 we consider the pairing susceptibility $P_{SC}(q = 0, i\omega_m = 0)$ and the current-current correlation function $\langle J_z J_x \rangle$. The even sector contribution (green curve) does not match the ED result (red curve). The difference is exactly compensated by the odd sector contribution (blue curve).
phase and the deconfined Dirac phase. According to the Mermin-Wagner theorem, a continuous symmetry cannot be spontaneously broken in two dimensions at finite temperature. Therefore, in order to reliably probe zero temperature ordering, we must first extrapolate the QMC data first to zero temperature and only after to the thermodynamic limit, $L \to \infty$.

To illustrate this procedure, in Fig. S6a, we depict the s-wave pairing order parameter, $\Delta_{SC}(L, \beta)$, for a range of system sizes $L = 6, 8, 10, 12$ and inverse temperature $\beta = L, L + 4, L + 8$. We observe that the results converge, with in the error bars, to the zero temperature limit.

Next we use the QMC data obtained at the lowest temperature considered, $\beta = L + 8$, to extrapolate to the thermodynamic limit. This is shown in Fig. S6b, where we fit finite system size data to the functional form $\Delta_{SC}(1/L) = a + b/L$.

Next we use the QMC data obtained at the lowest temperature considered, $\beta = L + 8$, to extrapolate to the thermodynamic limit. This is shown in Fig. S6b, where we fit finite system size data to the functional form $\Delta_{SC}(1/L) = a + b/L$.

VI. THE $\pi$-FLUX LATTICE

The hopping Hamiltonian of the $\pi$-flux lattice is given by,

$$\mathcal{H} = - \sum_{r,r'} t_{r,r'} c^\dagger_r c_{r'} + h.c.,$$

where $t_{r,r'} = t \left[ (-1)^{r_x} \delta_{r_x, r'_x + \hat{e}_y} + \delta_{r_x, r'_x + \hat{e}_x} \right]$. To diagonalize the Hamiltonian we first double the unit cell, such that the total flux in the enlarged unit cell equals $2\pi$. We arrange the fermion operators belonging to each unit cell in a two component spinor,

$$\Psi_R = \{ c^\dagger_{R,A}, c^\dagger_{R,B} \}$$

where the sub lattice $\{ R, A \}$ ($\{ R, B \}$) is defined by the set of lattice points $\{ r_x, r_y \} = \{ 2n, m \}$ ($\{ 2n + 1, m \}$). We

![FIG. S8: Dispersion relation of the $\pi$-flux lattice. The first Brillouin zone contains two Dirac nodes at $\vec{k} = \{ \pi/2, \pm \pi/2 \}$](Image)
now transform the Hamiltonian to momentum space,

\[ \mathcal{H} = -t \sum_k \Psi^\dagger(k) \begin{pmatrix} -2 \cos(k_y) & 1 + e^{i2k_x} \\ 1 + e^{-i2k_x} & 2 \cos(k_y) \end{pmatrix} \Psi(k) \]  

(S21)

where,

\[ c_{A/B}(k) = \sum_R e^{iR \cdot k} c_{R,A/B}, \]  

(S22)

and the first Brillouin zone is defined by the region \( \pi/2 \leq k_x \leq \pi/2 \) and \( \pi \leq k_y \leq \pi \). It is convenient to express the \( 2 \times 2 \) matrix kernel using the Pauli matrices \( \vec{\sigma} \) as,

\[ \mathcal{H} = -t \sum_k \Psi^\dagger(k) \vec{d}_k \cdot \vec{\sigma} \Psi(k) \]  

(S23)

where \( \vec{d}_k = \{1 + \cos(2k_x), \sin(2k_x), -2 \cos(k_y)\} \). The dispersion relation is then,

\[ \epsilon_k = \pm 2t \sqrt{\cos^2(k_y) + \cos^2(k_x)} \]  

(S24)

which contains two Dirac nodes at \( \vec{k} = \{\pi/2, \pm \pi/2\} \) with a Fermi velocity \( v_F = 2t \), see Fig. S8.

The ground state energy at half-filling is then obtained by integrating over the negative energy band in the first Brillouin zone which gives an energy \( E_{\pi-\text{flux}}/t = -0.958091 \) per site and per spin.