**Sharp tunnelling resonance from the vibrations of an electronic Wigner crystal**

![Diagram of wafer growth sheet and measurement setup]

**Fig. S1 | Wafer growth sheet and measurement setup.** a, Schematics of measurement setup. b-c, The MBE growth profiles for wafers used in the experiment; b, carbon-doped 2D hole quantum well wafer (pf-5-11-12.1), and c, silicon-doped 2D electron (pf-11-18-13) quantum well wafers. d-e, Filling factor vs. bias voltage is determined by integrating the capacitance (inverse compressibility) measurements.

### 0. Samples

The samples are GaAs/AlGaAs heterostructures, and grown by molecular beam epitaxy at Princeton university. Two versions of wafer, hole-doped with Carbon (pf-5-11-12.1) and electron-doped with Silicon (pf-11-18-13), were used in these measurements. Both are grown in (100) direction. The wafers have cold-growth spacer layers (grown at 440 °C, otherwise at 620 °C; See blue colored layers in the Fig.)
S1b and c) to help prevent dopant migration from the bottom electrodes into the barriers and QWs. This technique improved the quality samples over the ones used in previous studies1 when compared by measuring the broadening of TDOS at even filling factors as shown in Fig. S2a-b. The broadening is inversely proportional to the quantum lifetime of quasiparticles in the system2,3. We obtain the broadening of 0.15 meV for holes and 0.70 meV for electrons. From these numbers, we roughly estimate the transport mobility of the hole and electron samples to be $\sim 8.0 \times 10^5$ cm$^2$/Vs and $1.0 \times 10^6$ cm$^2$/Vs, respectively. Note that the quality of the hole sample is actually higher as the quantum lifetime is $\sim 4$ times longer. The wafers are specially designed for tunneling measurements by having a quantum well (14 nm for hole and 23 nm for electron wafer) separated by a tunneling and an insulating barrier from heavily doped electrodes. We define multiple mesas of 150 $\mu$m in diameter using photolithography, and make electrical contacts on top and bottom electrodes for a vertical tunneling device. All measured samples (3 hole mesas from pf-5-11-12.1 and 2 electron mesas from pf-11-18-13) showed the same resonance features, respectively, within our experimental resolution.

**Fig. S2 | The broadening of LL DOS as a measure of disorder strength.** a-b, TDOS of the hole and electron wafers measured at $v \sim 4$ and $T=25$ mK. The broadening of the TDOS near the Fermi level is proportional to the inverse of the quantum lifetime and due to disorder in the systems. c, The increase of capacitance of samples as hole and electrons are induced into the 2D QWs. The sharper capacitance increase means less disorder in the system. The quality of the wafers in this study (red curves) is significantly improved over the one in Dial et. al. *Nature* 464, 566–70 (2010) (blue curve).
1. Calculation of tunneling matrix modification due to bosonic mode

The differential tunneling conductance $G$ (or tunneling density of states; TDOS) is written as in the zero temperature limit,

$$G = dI/dV_t = \int \int d^2 k |T(\vec{k}, E_F + eV_t)|^2 A(k, E_F + eV_t),$$

where $T(\vec{k}, E)$ is the tunneling matrix element, and $A(k, E)$ is the spectral function of the 2D system of interest. The change in conductance can be categorized by (i) the symmetry of signal in bias voltage and

(ii) the existence of the energy loss in the tunneling process as follows (see also Fig. S3):

(i) The symmetry of the self-energy function dictates that the real part of $\Sigma$ generates an anti-symmetric (odd) signal in the tunneling conductance while the imaginary part produces a symmetric (even) change in conductance.

(ii) The inelastic tunneling involves the emission of real phonons, whereas the elastic conductance contribution is attributed to the emission and re-absorption of virtual phonons.

Thus, the signatures of electron-phonon (boson) coupling in the electron tunneling spectrum are a combination of peak/dip or step/anti-step features located at $\pm(\Delta_g + \epsilon_{vH})$ in $dI/dV$, where $\Delta_g$ is the single-particle energy gap near the Fermi energy. As noticed by many authors $^{4-7}$, the change in the tunneling matrix element can be large when the ratio $\Sigma/E_F$ is not small.

Fig. S3 | Tunneling conductance modification due to bosonic modes (phonons). a–c, Diagrams show the electrons interact with phonons (wavy lines) in the process of tunneling from left to right electrodes (blue) in the case of $\Delta_g = 0$. Occurrence of symmetric and anti-symmetric features in the tunneling conductance $G$ depends on the number of electron-phonon scatterings whose strength are given by $g$. Direct (a), inelastic (b), and elastic (c) tunneling mechanisms are displayed.
The self-energy term \( \Sigma(E) \) modifies the tunneling matrix element because \( \Sigma(E) \) acts as a complex-valued energy-dependent potential in the 2D quantum well (which we refer to as the right electrode). The argument is roughly based on the work of Taylor et. al.\(^6\) (For more complete calculations, see Davis et. al.\(^4\) and Appelbaum et. al.\(^5\)). We give a simple estimate of the change of the tunneling matrix \( T(\vec{k}, E) \), taking account of the dimensionality of electrodes and experimental parameters.

Assuming the effective masses \( m_e \) of two electrodes are the same, the momentum and energy conservation law of 3D (left)-2D (right) planar tunneling leads to the following selection rules:

\[
k_{\parallel l} = k_{\parallel r}, \quad \frac{\hbar^2}{2m_e} (k_{\parallel l}^2 + k_{\perp l}^2) + E_{l0} = \frac{\hbar^2}{2m_e} k_{\parallel r}^2 + E_{r0} + \Sigma. \]

These lead to \( \frac{\hbar^2}{2m_e} k_{\perp l}^2 = \Delta E_0 + \Sigma \). Here, \( k_l \) (\( k_r \)) is the wave vector in the left (right) electrode, and the perpendicular (parallel) components to the barrier interface are indicated with suffix \( \perp(\parallel) \). Also, \( E_{l0} \) (\( E_{r0} \)) is the conduction band edge on the left (right) electrode, and the conduction band edge difference is \( \Delta E_0 = E_{r0} - E_{l0} \). The non-zero self-energy \( \Sigma \) in the right electrode modifies the selection of the perpendicular wave vectors \( k_{\perp l} \) of the left electrode as follows,

\[
k_{\perp l} = \frac{1}{\hbar^2} \sqrt{2m_e (\Delta E_0 + \Sigma)}, \quad k_{\perp l} \approx 1 + \frac{\Sigma}{2 \Delta E_0}, \]

where \( k_{\perp l} \) is the wave vectors on the left electrode with the self-energy term being zero. We insert \( k_{\perp l} \) into the expression of the tunneling matrix element for a rectangular tunneling barrier\(^7\):

\[
|T(\vec{k}, E)|^2 = \frac{\kappa^2 \text{Re}(k_{\perp l}) \text{Re}(k_{\perp r}) e^{-2\kappa d}}{|\kappa - ik_{\perp l}|^2|\kappa - ik_{\perp r}|^2},
\]

where the barrier thickness is \( d \), the tunneling exponent is \( \kappa = \frac{1}{\hbar} \sqrt{2m(U - E)} \), and \( U \) is the tunneling barrier height. Also, note that in a quantum well \( k_{r\perp} = \sqrt{2mE_{c0}/\hbar} \), with \( E_{c0} \) being the lowest confinement energy of the right electrode. Then, we can approximate the tunneling matrix change as

\[
\frac{|T(\vec{k}, E + \Sigma)|^2}{|T(\vec{k}, E)|^2} \approx 1 - \frac{1}{2} \frac{\kappa^2 - k_{\perp l0}^2}{\kappa^2 + k_{\perp l0}^2} \frac{\text{Re} \Sigma}{\Delta E_0} + \frac{\kappa k_{\perp l0}}{\kappa^2 + k_{\perp l0}^2} \frac{\text{Im} \Sigma}{\Delta E_0}. \]

Note that \( \Delta E_0 \) is merely the difference of the Fermi energies between left and right electrodes, \( \Delta E_0 = (E_F - E_{l0}) - (E_F - E_{r0}) = (E_{lF0} - E_{rF0}) = \Delta E_{F0} \), where \( E_F \) is the Fermi level, and \( E_{F0} \) is the Fermi energy defined as the energy difference between the highest and the lowest occupied states.

As a result, the changes of differential conductance \( G \), normalized to the conductance without electron-phonon interaction \( G_0 \), are expressed as including the inelastic contribution \( G_0 \):
where $l_i$ is the phenomenological effective inelastic scattering length\(^6\). Note that the inelastic contribution is symmetric in tunneling energy, and the elastic part is a combination of anti-symmetric and symmetric component (Fig. S3).

Fig. S4 | Tunneling conductance modification due to bosonic modes (phonons). a-b, Cartoon pictures that represent a tunneling electron interacting with a phonon of energy $h\Omega_{ph}$ (wavy arrows) at positive (a) and at negative bias (b). The right electrode visualizes an energy gap $\Delta_g$ near $E_F$ in the single-particle density of states of the 2D electron system under magnetic fields. When a real phonon is emitted, the tunneling process is inelastic. When a virtual phonon is emitted and reabsorbed, the tunneling is elastic. In either case, the influence of the electron-phonon interaction to tunneling signal is most pronounced when $\alpha = \pm (\Delta_g + h\Omega_{ph})$, because the states above (below) this bias energy at positive (negative) bias have a probability to emit a phonon and decay into the bottom (top) of the available states. c-e, Simulations of TDOS with e-ph coupling with $\alpha^2 F$ of an Einstein phonon mode of $h\Omega_{ph}$ and constant $N(\omega)$. No gap (c), symmetric gap (d) and asymmetric gap (e) are simulated. Note the resonant features are located at $\Delta_g + h\Omega_{ph}$. 

\[
\frac{G_{\text{inelastic}}}{G_0} = -k_F l_i \frac{\text{Im}\Sigma}{\Delta E_{F0}} \\
\frac{G_{\text{elastic}}}{G_0} = -\frac{\kappa^2 - k_{LL,0}^2}{2\kappa^2 + k_{LL,0}^2} \frac{\text{Re}\Sigma}{\Delta E_{F0}} + \frac{\kappa k_{LL,0}}{\kappa^2 + k_{LL,0}^2} \frac{\text{Im}\Sigma}{\Delta E_{F0}},
\]
In our experiment, the shape of the observed tunneling conductance strongly suggests that the elastic tunneling process is dominant over the inelastic one. We can understand this as follows. The elastic tunneling involves the two successive scatterings (emission and reabsorption) with a virtual-phonon while the inelastic tunneling process induces a single real-phonon scattering (emission). Compared to the elastic process coupled with a short-lived virtual phonon, the inelastic tunneling leads to the generation of a real phonon ($k_{\text{ph}} \sim 0.01 \ \text{Å}^{-1}$) that requires a more strict condition for the conservation of energy and momentum, and is often strongly suppressed in high quality planar tunneling junctions. Thus, one needs the phenomenological effective inelastic scattering length $l_i$ in Eq. (1) to take this into account. In our devices, the planar momentum is well conserved within $0.001 \ \text{Å}^{-1}$, leading to the suppression of inelastic processes. The fact that the phonon is located inside one of the electrodes, not at the barrier, additionally disfavors the traditional inelastic processes\textsuperscript{6,8}.

Then, the dominant tunneling process is basically the self-energy effect\textsuperscript{4,7}. Based on Eq. (2) and $k_{\perp 0} \ll \kappa$ from our experimental parameters, the dominant effect comes from the real part of the self-energy ($\text{Re} \ \Sigma$) that generates an odd (anti-symmetric) logarithm singularity feature in the $dI/dV$ spectrum. Thus,

$$G = \left(1 - \frac{1}{2} \frac{k^2}{k^2 + k_{\perp 0}^2} \frac{\text{Re} \Sigma}{\Delta E_{F0}} \right) G_0,$$

(3)

And,

$$\text{Re} \ \Sigma = - \frac{G - G_0}{G_0} \frac{k^2}{k^2 - k_{\perp 0}^2} \frac{2 \Delta E_{F0}}{2 \Delta E_{F0}}.$$

(4)

Our estimates of the Fermi energies of a hole-doped sample are $E_{F0} \sim 3 \ \text{meV}$ on the left 3D electrode and $E_{F0} \sim 2.4 \ \text{meV}$ on the right 2D electrode when density of QW is $n_{2D} = 2.0 \times 10^{11} \ \text{cm}^{-2}$ ($\nu \sim 1$ at 8 T). Thus, $\Delta E_{F0} = E_{F0} - E_{F0} \sim 0.6 \ \text{meV}$, and the peak value of Re $\Sigma$ becomes as large as 0.4 meV which can account for the 30% change of tunneling conductance observed in our measurement.

Additionally, this model correctly predicts that the strength of the resonance features becomes bigger as the 2D QW density increases, because the value of $\Delta E_{F0} = E_{F0} - E_{F0}$ is getting smaller (note $E_{F0}$ is linearly proportional to density of the QW); we indeed observe that the e-ph signal strength is stronger at $\nu = 1$ compared to $\nu = 0$ for hole samples. For more accurate estimate, however, a theoretical treatment including the effect of magnetic fields with Landau quantization may be necessary.

### 2. Numerical simulation of tunneling conductance

The self-energy $\Sigma$ is related to the Eliashberg function, $\alpha^2 F(\omega)$, where $F(\omega)$ is the phonon density of states, and $\alpha = \alpha(\omega)$ is the phonon coupling strength, and expressed as follows in the limit of $T = 0$,

$$\Sigma(\omega) = \frac{A}{N_0} \int_0^\infty d\omega' \alpha^2 F(\omega') \int_{-\infty}^\infty d\omega'' N(\omega'') \left[ \frac{\theta(\omega'')}{\omega + \omega' - \omega'' + i0^+} + \frac{\theta(-\omega'')}{\omega - \omega' - \omega'' + i0^+} \right],$$

(5)
where $\alpha^2 F(\omega)$ is the Eliashberg function, $N(\omega)$ is the electronic density of states, and $i0^+$ is an arbitrarily small imaginary number. In principle, we can experimentally determine $\alpha^2 F(\omega)$ by inverting this equation from $\Sigma$ obtained from Eq. (4). However, the inversion of the integral equation has long been known to be a mathematically unstable procedure. Instead, we can simulate the $\Sigma$ by modeling $N(\omega)$ and $\alpha^2 F$ with a sum of Lorentzian distributions, evaluate Eq. (5) and plug the calculated $\Sigma$ into Eq.(3), and compare with real data. For this, we call this effective inversion procedure the tunneling conductance “simulation” (See Fig. S4 and S5).

Fig. S5 | Simulation of tunneling conductance and Eliashberg function. a. By simulating with a broadened Eliashberg function $\alpha^2 F$ and density of states $N(\omega)$, the self-energy $\Sigma$ and the tunneling conductance are obtained. Equations of (3-5) are used. b-c, Reproduction of Fig.1e for comparison with the simulation. The normalized conductance $\frac{\sigma(\hbar \omega) - \sigma_0}{\sigma_0}$ displays close resemblance to the simulated $Re \Sigma$ in a. Also compare the Raw TDOS in c to the TDOS (dark green) in a.

For evaluating the Eq. (5), we need to know or assume $\alpha^2 F$ and $N(\omega)$. The examples in Fig. S4e-e are demonstrated by assuming an Einstein phonon mode (a delta function distribution) at $\hbar \omega = \varepsilon_{\text{FH}}$ for $\alpha^2 F$, and constant $N(\omega)$ with an arbitrary energy gap $\Delta_g \approx$ near Fermi level. The most important aspect is that the resonance features are located at energies $\varepsilon_r = \Delta_g + \varepsilon_{\text{FH}}$. A simulation with more realistic $N(\omega)$ with broadening and an asymmetric structure in energy and $\alpha^2 F$ with a Lorentzian broadening is shown in Fig. S5, in comparison with experimental data.
3. Estimation of the lattice correlation length

We estimate the lattice correlation length $L_c$ by measuring the statistical broadening of the energy of van Hove singularity. From the relationship $\epsilon_{vH} \propto 1/l_s^3$, we have $\Delta \epsilon_{vH}/\epsilon_{vH} = \Delta \left( \frac{1}{l_s^3} \right)/\frac{1}{l_s^3}$, and $\Delta \epsilon_{vH}/\epsilon_{vH} = -3 \times \Delta l_s/\bar{l}_s$. Thus,

$$L_c \sim (\Delta l_s)^{-1} \frac{\epsilon_{vH}}{\bar{l}_s \Delta \epsilon_{vH}}$$

(6)

where $\Delta \epsilon_{vH}$ is the statistical distribution of the van Hove singularity energy, $\Delta l_s$ is the statistical distribution of lattice spacings, $\bar{l}_s \sim \sqrt{2\pi l_B/\sqrt{\nu}}$ is the mean value of $l_s$, $l_B$ is the magnetic length, and $L_c$
is the lattice correlation length.

We can estimate $\varepsilon_{vH}$ and $\Delta\varepsilon_{vH}$ from the location and broadening of $\alpha^2F$ which strongly peaks at $\varepsilon_{vH}$. The Eliashberg function $\alpha^2F$ can be estimated by performing a tunneling conductance simulation. We model a simple $N(\omega)$ with a sharp gap edges at $\Delta_g$, and $\alpha^2F$ with a sum of several Lorentzian distributions. Then, with the $N(\omega)$ and $\alpha^2F$, we can calculate $Re\Sigma$ using Eq. (5). This calculated $Re\Sigma$ is compared with measurement data for a least square regression fit process. The fitting parameters are locations, widths, and heights of Lorentzian distributions of $\alpha^2F$ (see Fig. S6c-f for the results), so the number of parameters of the fit are $3 \times$(number of Lorentzians). In Fig. S6, two Lorentzians are used in (f), and one is used in (h). The curve fit is slightly better with two Lorentzians, and the difference in estimation of the broadening of $\alpha^2F$ is only ~10 percent. Thus, we chose the value obtained from the fit using two Lorentzians. Finally, we would equate the energy broadening of $\alpha^2F$ to $\Delta\varepsilon_{vH}$, and calculate the distribution of lattice parameter $\tilde{\ell}_s$ from Eq.(6).

By fitting the self-energy and obtaining the broadening of the modeled Eliashberg function $\alpha^2F$, we get $\Delta\varepsilon_{vH}/\varepsilon_{vH} \sim 20\%$. Thus, we estimate $\Delta\ell_s/\tilde{\ell}_s \sim 7\%$ and the crystal correlation length $L_c \sim 15\tilde{\ell}_s$. Because $\tilde{\ell}_s$ at $B_\perp = 8T$ and $\nu = 0.88$ (i.e. $\nu_q = 0.12$) is about 65 nm, this corresponds to $L_c = 15 \times 65\,\text{nm} \sim 970\,\text{nm}$.

We performed the same estimation for the electron system (Fig. S6h-i). With $B_\perp = 5.8T$ and $\nu = 1.1$ (i.e. $\nu_q = 0.1$), the quasiparticle lattice spacing is $\tilde{\ell}_s \sim 85\,\text{nm}$. Also, from the Eliashberg function fit we get $\Delta\varepsilon_{vH}/\varepsilon_{vH} \sim 85\%$, it gives $\Delta\ell_s/\tilde{\ell}_s \sim 30\%$. Thus, it is about 2.5~3 lattice spacing of correlation with the correlation length $L_c \sim 3 \times 85\,\text{nm} \sim 250\,\text{nm}$.

It is important to note that this estimation will give only the absolute lower bound of the lattice coherence length $L_c$; $\alpha^2F$ has an intrinsic width in energy (even in a perfect crystal), i.e. $\Delta(\alpha^2F)_{\text{observed}} = \Delta\varepsilon_{vH} + \Delta(\alpha^2F)_{\text{intrinsic}}$, and $\Delta\varepsilon_{vH}$ is always smaller than $\Delta(\alpha^2F)_{\text{observed}}$. Thus, by estimating the actual disorder-induced broadening $\Delta\varepsilon_{vH}$ from $\Delta(\alpha^2F)_{\text{observed}}$ from the simulation, we would get a value for $L_c$ that is smaller than the actual value.

### 4. Details of curve fits for $\varepsilon_r$ on $\nu^{3/2}$

In Fig.2 and Fig. S7a, we show curve-fits of the measured resonance energy $\varepsilon_r$ to the functional form of $\nu^{3/2}$ (referenced to the Coulomb gap $\Delta_c$) in various magnetic fields. The location of the resonance $\varepsilon_r$ as a function of filling factor $\nu$ is fitted with curves of the following form:

$$
\left( \frac{e^2}{4\pi\varepsilon_s l_B} \right) \left( C_1 + C_2 \left( \frac{\sqrt{3}\nu}{\pi} \right)^{3/2} \right) = \Delta_g + \varepsilon_{vH}.
$$

Two field-independent fit parameters $C_1$ and $C_2$ are used to fit all field data. The location of the gap edge ($\Delta_g$) is determined by locating a peak in $d^2I/dV_t^2$ spectrum, which corresponds to the sharp increase in $dl/dV_t$ spectrum. And, the location of the resonance is obtained by fitting with the local extrema in $dl/dV_t$ data.

The horizontal dotted yellow lines in Fig. S7a represent the magnetic-field-dependent Coulomb gap $\Delta_g = \Delta_c = (e^2/\hbar l_B)C_1$, and the red dotted curves indicate the fit curves for $\varepsilon_{vH} = (e^2/4\pi\varepsilon_s l_B)(\sqrt{3}\nu/\pi)^{3/2}C_2$. The fact that we have only two free parameters $C_1$ and $C_2$ throughout the all
magnetic fields data indicates that the curves are functions of only \( E_c = e^2/4\pi\varepsilon_s l_B \) and \( \nu \) regardless of magnetic fields. The fitting of the semi-classical formula (blue dashed lines) is done in the range \( 0.9 < \nu < 1 \), where the fitting parameters are \( C_1 = 0.0165 \) and \( C_2 = 0.093 \). The deviations at \( \nu < \sim 0.88 \) are obvious.

The curves with an additional quantum mechanical effect included (red dotted lines) are based on the shear modulus of the composite fermionic crystals in the lowest Landau level calculated by the quantum Monte-Carlo simulation of Archer et. al.\(^9\). We estimated the energies of the phonon modes by assuming that they are proportional to the square-root of the shear modulus, \( \sqrt{C_t} \); we believe this is a qualitatively reasonable assumption beyond semi-classical limit. The composite fermionic crystal with 4 flux quanta attached is the lowest energy wave function at \( 0.12 < \nu_{qp} < 0.18 \), and still very close to the ground state even below \( \nu_{qp} < 0.12 \). The numbers beyond the applicable region of the simulation (i.e. \( 0.18 < \nu_{qp} > 1/6 \)) were obtained by extrapolation, which is semi-quantitatively validated from the behavior of the shear modulus in this regime from the analytic solution of the same problem\(^10\). The theories also show that near \( \nu_{qp} = 0.18 \) the crystal of 4-CF (composite fermions with 4 flux quanta attached)
will go through a transition to the crystal of 2-CF (one with two flux quanta attached) before 1/5 FQHE is
reached. This theoretical prediction appears to be consistent with the filling factor where the tunneling
resonance feature disappears. The strong increase of stiffness of the crystal shown in the simulation can
explain the resonance energy changes near \( v_{qp} \sim 0.18 \).

5. Interpretation of the offset energy near \( v = 1 \)

The resonances display small energy offsets involved in injection (for \( v < 1 \)) and ejection (for \( v > 1 \)) spectra (See orange arrows in Fig. 1f). The qualitative behavior of this observed signal is
consistent with the quasiparticles interacting with the magnetophonons. A naive expectation would be that
the magnetophonon energy \( \epsilon_{\nu H} \) goes to zero (and thus \( \epsilon_r \rightarrow \Delta_C \)) as the quasiparticle density vanishes
approaching \( v = 1 \) (i.e. \( v_{qp} \rightarrow 0 \)). However, since the scattering of the virtual (magneto)phonon is a spin-
 preserving process, the high energy quasiparticles can only relax into the states with same spin (in the
short time scale allowed for the virtual process). Around \( v = 1 \), the exchange enhanced spin-gap exists
and is expected to be the main source of the energy offset generating the offset asymmetry of the resonant
features. Specifically, in the case of injecting particles with minority spins, the lowest lying available
states are located higher than Fermi level by the spin exchange gap \( \Delta_s \) near \( v = 1 \).

In both cases of the hole system and the electron system with a large in-plane field, our view is
that the Wigner crystal is very nearly fully or fully spin-polarized for filling factors near \( v=1 \). As shown
below in the diagram, there are four possible cases of injections and ejections (injection of “anti-particle”;
we use this term instead of “hole” to avoid possible confusion to positively charged doped carriers),
where the tunneling particle goes to/come from interstitial space; the observed resonances are consistent
with these cases.

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**Fig. S8 | Spin configurations near \( v = 1 \).** The offset of the resonance features suggest that injecting with spin-
up particles show the resonance experiencing the additional spin energy (spin-gap), while the injecting with spin-
down particles do not generate resonance possibly because the particles are force to fill the crystal lattice sites.
We take the example of $\nu < 1$ (we believe these arguments to apply to both $\nu < 1$ and $\nu > 1$). For $\nu < 1$ the bulk has spin-down particles and the vacancies that comprise the Wigner Crystal, which we have labeled with white spin-down arrows. The Fermi level is pinned to the spin down states as these are the states as this is the spin state being filled for $\nu < 1$. When a particle is ejected, we then expect it to be a spin-down particle. Upon injecting a particle, there are two possibilities. There are vacancies in the spin-down background that could be filled with spin-down particles, and there are sites at higher energy that could be filled with spin-up carriers. The data show a shift-up energy for injection, indicating that the observed signal arises from injecting spin-up carriers. We believe this arises simply because there are many more spin-up sites to fill than spin-down sites (the spin-down sites for injection are the vacancies (lattice sites) that comprise the Wigner Crystal).

The measurement of the asymmetry of the locations of the features above and below Fermi level will give information about the spin gap near $\nu = 1$. As shown in Fig. S7b, the spin gap energy can be extracted as the separation of two locations ($|U| - |L|$). Surprisingly, we find that the spin-gap decreases as we approach $\nu = 1$. This may be a feature of hole-doped samples that likely contain substantial Landau level and subband mixing.

6. Magnetophonon, magnetoplasmon and mass-independence

A two-dimensional electron crystal has one transverse and one longitudinal phonon vibrational modes in the absence of the magnetic field. When a perpendicular magnetic field is applied, the two modes are hybridized into magnetophonon and magnetoplasmon because the electrons experience the Lorentz force and circulate into cyclotron orbits. The magnetophonon is the gapless Goldstone mode while the magnetoplasmon is gapped with the cyclotron energy. In this section, we give a brief summary of the modes and show that the energy scaling law of the magnetophonon is proportional to $n^{3/2}/B$.

The displacement of the electron $r_i$ at the lattice point $R_i$ is expressed in its Fourier component $r_k$ as

$$ r_i = \frac{1}{\sqrt{N}} \sum_k r_k e^{i \mathbf{k} \cdot \mathbf{R}_i}. $$

In an oscillatory motion $r_k(t) = r_k e^{i \omega_k t}$, the equation of motion in the absence of magnetic field is given as,

$$ -\omega_k^2 r_k + \hat{C}(\mathbf{k}) r_k = 0. $$

Here, $\hat{C}(\mathbf{k})$ is the dynamical matrix which has the information about the Coulombic restoring force of the system. After solving the equation by diagonalizing the dynamical matrix $\hat{C}(\mathbf{k})$, we can express the equation of motion with a new diagonal matrix $\hat{D}(\mathbf{k})$ in new basis vectors $u_k$,

$$ -\omega_k^2 u_k + \hat{D}(\mathbf{k}) u_k = 0, $$

$$ \hat{D} = \begin{pmatrix} \omega_{\nu k}^2 & 0 \\ 0 & \omega_{\nu L k}^2 \end{pmatrix}. $$
Here, $\omega_{TR}$ ($\omega_{L,k}$) is the transverse (longitudinal) mode frequency. The full solutions are given by Bonsall et al.\cite{12}, and the overall energy scale of the modes are proportional to $\omega_0 = (8e^2/\hbar^3 m)^{1/2}$.

Our interest is the case with a perpendicular magnetic field. In a magnetic field, the Lorentz force couples the two otherwise independent modes, and the new equation of motion is\cite{13}

$$-\omega_k^2 u_k + i \omega_k \omega_c u_k \times \hat{z} + \vec{D}(\vec{k}) \vec{u}_k = 0. \quad (7)$$

The non-trivial solutions of the equation of motion should satisfy

$$\begin{vmatrix}
\omega_T^2 - \omega_k^2 & i \omega_c \omega_k \\
-i \omega_c \omega_k & \frac{\omega_L^2}{\omega_k^2} - \omega_k^2 \\
\end{vmatrix} = 0.$$

The two solutions $\omega_-$ and $\omega_+$ are called magnetophonon and magnetoplasmon, respectively, and $\omega_c$ is the cyclotron frequency. In the limit of $\omega_c \gg \omega_T, \omega_L$, $\omega_-(\vec{k}) = \frac{\omega_T \omega_L}{\omega_c} = \left(\frac{e^2}{\hbar l_B}\right)\left(\frac{\sqrt{3} \pi}{\hbar}\right)^2 \sqrt{\text{det}[\vec{D}(\vec{k})]} \right)$. 

**Fig. S9** | Log-Log plot of the power law $n^{3/2}$ and mass independence of phonon dispersion. a-b, A comparison of the energy scale of electron and hole samples at 7 T. The energy referenced to the gap $\Delta_g$ is displayed as a function of filling factor. The dotted lines are guides to the eye and separately plotted in b for a direct comparison. c, The locations $E_{qp}$ of the resonance features of electrons ($B_\perp = 5, 6, 7, 8$ Tesla; magenta markers) with in-plane fields and those of holes ($B_\perp = 6, 7, 8$ Tesla; green markers) are plotted as a function of density. d, $E_\perp - \Delta_g$ is divided by $\sqrt{B}$ as a Coulomb energy scale and plotted against $v_{qp}$. Note all data points collapse onto the predicted $v^{3/2}$ curve. e, The same data are displayed as a log-log plot. After taking account of the 17% difference in magnitude of electron and hole data (a shift of -0.08 of electron data in the log plot), the least square fit of the data in $0.05 < v_{qp} < 0.11$ (indicated by blue circles) shows the power exponent of $1.68 \pm 0.081$, which demonstrates a reasonably good agreement with the power law of 3/2.
\[ \omega_+ (\mathbf{k}) = \omega_c + \frac{\omega_T^2 + \omega_L^2}{2\omega_c} = \omega_c + \frac{1}{2} \left( \frac{e^2}{\hbar l_B} \right) \frac{\sqrt{3}}{\pi} \text{tr}[\tilde{D}(\mathbf{k})], \]

showing the magnetophonon remains gapless while the magnatoplasmon is gapped by \( \omega_c \). Here, \( \tilde{D}(\mathbf{k}) = (m l_3^2 / 8 e^2) \tilde{D}(\mathbf{k}) \) is a dimensionless dynamical matrix whose magnitude is an order of unity \(^{14}\).

The energy scaling laws of both of \( \omega_T k \) and \( \omega_L k \) in density tuning (\( n \sim vB \sim 1/l_3^2 \)) are \( \omega_0 \sim (8 e^2 / m l_3^2)^{1/2} \); thus, the magnetophonon is scaled as \( \omega_+ \sim 1/l_3^2 B \sim n^{3/2} / B \sim \sqrt{B} v^{3/2} \). This power law is the distinctive feature of the magnetophonon spectrum.

Probably the most remarkable feature of the scaling law of the magnetophonon is its carrier mass independence\(^{11}\) (see Fig. S8). In the large field limit, the Lorentz force dominates the mass inertia, and the motion is defined by the balance between the Lorentz force and the Coulombic restoring force; the mass term in the equation of motion, Eq. (7), becomes negligible. In Fig. S8, we compare the spectra of electron and hole samples to find that the magnetophonon energy scale is comparable in magnitude even though the effective mass of holes is \( \sim 6 \) times heavier than that of electrons.

7. Analysis of the “vertical features” in tunneling current \( I_t \) and \( dI/dV \)

The seemingly vertical features in data (see arrows in Fig. S11c and also in Fig. 1f) are consequences of an abrupt disappearance of the modified tunneling matrix element by electron-phonon coupling at the phase boundary. The increased noise suggests the existence of small hysteresis due to the first-order phase transition.

The tunneling current \( I \) measured at bias voltage \( V_t \) has the contributions from all electrons tunneling at energies between \( eV_t \) and the Fermi level (\( eV_t = 0 \)) (see Fig. S10a). In other words, the tunneling current \( I \) is the summation of \( dI/dV \) up to the bias energy \( E_t = eV_t \), and given as (at \( T = 0 \))

\[
I = \int_0^{eV_t} e(dV_t) \frac{dI}{e dV_t} = \int_0^{eV_t} dE_t \int_{-\infty}^{\infty} d^2 k \left| T(\mathbf{k}, E_F + E_t) \right|^2 A(\mathbf{k}, E_F + E_t),
\]

**Fig. S10 | Effect of tunneling matrix element change.** a, Tunneling current with a modified tunneling matrix element at energy \( E_t = \varepsilon_r \). b, Tunneling conductance and tunneling current with/without electron-boson coupling (orange/dotted). While the effect of a peak (or a dip) in the tunneling matrix only affects \( dl/dV \) in the vicinity of the peak, the total tunneling current for injection voltages beyond the peak is kept increased from the bare value. If the peak suddenly disappears as the filling factor changes, it induces spurious vertical noise in otherwise smooth \( dl/dV \) as shown by arrows in Fig. S11c.
and it is apparent that a change in $dI/dV_t$ (or $T(k, E)$) at certain tunneling bias energy $E_t = \epsilon_r$ should affect the measured current $I$ at subsequently higher bias voltages (see Fig. S10b). On the other hand, this is supposed to affect the differential conductance $dI/dV_t$ only around $E_t = \epsilon_r$. However, when there is an abrupt change in a component of the tunneling matrix while varying filling factor $\nu$, a big change in $I (E_t > \epsilon_{\nu H})$ can induce large noise in $dI/dV_t$, extending vertically at around the filling factor the tunneling matrix change occurs (See arrows in Fig. S11c).

Moreover, the existence of hysteresis in changing filling factor $\nu$ seems essential in having this noise pronounced. Even though the filling factor $\nu$ is strictly fixed along the vertical axis in the data presented, such as Fig. S11c, due to the nature of our feed-back based measurement scheme, there are intermediate measurement steps where the filling factor deviates from a desired value by some degree. If a small hysteresis exists, near the phase boundary, the intermediate feed-back steps will make the system end up randomly in either phase (either liquid or crystal) through uncontrolled phase transitions. This will induce large noise, again situated vertically (at around certain value of $\nu$) in the 2D image data of $dI/dV_t$ as shown in Fig. S11b and c. Note that without hysteresis, regardless of the intermediate steps, the phase of the system is well decided by the values $eV_t$ and $\nu$ of the final measurement step, and the noise will disappear. Note that in the limit of large hysteresis this noise will also disappear because the phase of the electron system is effectively fixed. Thus, $dI/dV_t$ image data plotted in ($eV_t$, $\nu$) with vertical lines of increased noise is a consequence of a “small-size” hysteresis belonging to a first-order phase transition at that filling factor.

**Fig. S11 | Origin of the vertical features: First order phase transition.** a-b, Diagrams showing $I$ and $dI/dV$ in the case of sudden disappearances of phonons at the phase transition. TDOS (or $dI/dV$) can show spurious noise due to a sudden change in tunneling matrix $T_k$ (and possible hysteresis) at tunneling bias $eV$ near Fermi level. The noise will be greatly amplified with the existence of hysteresis in the phase transition. The existence of $T'_k$ whose value is suppressed (or enhanced) from $T_k$ will determine the regions separated by the dotted vertical line in b.
8. Additional data plots (Figs. 12-14)

**Fig. S12** | **Resonance features at even filling factors**

- **a**, Normalized TDOS of a hole system at 8 T.
- **b**, TDOS and **c**, Background-subtracted TDOS of a hole system at 5 T at T=25 mK.

(Continues on next page)
Fig. S13 | Temperature dependence of the resonances. a-b, Background subtracted conductance $dl/dV_t$ for the hole doped sample at $B_\perp = 8 \, T$ at 25 mK and 250 mK. c-d, Background subtracted current $I$. The yellow circles in a emphasize that the energetics of the phonon features near $\nu \sim 0$ is particle-hole symmetric to ones near $\nu \sim 1$. e-g, TDOS of 2D electrons with $B_\perp = 5.8 \, T, B_\parallel = 8.2 \, T$ measured at 25 mK (e), 300 mK (f) and 500mK (g). Spectra at e, f and g are after being subtracted with slowly-varying backgrounds to reveal fine structures of the spectrum. Raw data are shown in Fig. S14b. The colored circles highlight features of magnetophonon disappearing with increasing temperature.

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Fig. S14 | The $dI/dV_t$ spectrum of electron-doped sample. a-b, The raw spectra of Fig. 4 are plotted before being subtracted with a slowly-varying background, with $B_\perp = 6\, T, B_\parallel = 0\, T$ (a), and $B_\perp = 5.8\, T, B_\parallel = 8.2\, T$ (b). c-e, Background subtracted TDOS of 2D electrons with $B_\perp = 5.8\, T, B_\parallel = 8.2\, T$ is measured up to second Landau level at 25 mK (c), and 500 mK (d). The colored arrows highlight features existing only at low temperatures. In e, the spectrum in c is subtracted with the one in d to show strongly temperature dependent features shown inside squares.

References


