Universal dynamics and deterministic switching of dissipative Kerr solitons in optical microresonators

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ANALYTICAL APPROXIMATION OF SOLITONS

We start from the master equations for the internal \( A(\phi, t) \) and external \( s_{\text{in/out}} \) fields, where \( \phi \) is the angular coordinate in a ring resonator and \( t \) is the slow time (slower than the roundtrip) \([1, 2]\):

\[
\frac{\partial A}{\partial t} - i \frac{D_2}{2} \frac{\partial^2 A}{\partial \phi^2} - ig |A|^2 A + (\kappa/2 + i \Delta) A = \sqrt{\eta} s_{\text{in}},
\]

\[
s_{\text{out}} = s_{\text{in}} - \sqrt{\eta} A.
\]

Here the input continuous wave (CW) pump \( P_{\text{in}} = \hbar \omega_0 |s_{\text{in}}|^2 \), and mean output power \( P_{\text{out}} = \hbar \omega_0 \frac{1}{2\pi} \int |s_{\text{out}}|^2 d\phi \).

\[\eta = \kappa_{\text{ex}} / \kappa \] is the coupling efficiency determined as a ratio of resonator-bus waveguide coupling \( \kappa_{\text{ex}} \) to total resonator losses \( \kappa \) (intrinsic loss and coupling loss), with critical coupling corresponding to \( \eta = 1/2 \), \( \hbar \) is reduced Planck’s constant, \( \omega_0 \) is the pumped optical frequency. \( D_2 \) describes the second order dispersion in the Taylor series for eigenfrequencies of azimuthal whispering gallery modes around the pumped mode \( \omega_0 \): \( \omega_{\mu} - \omega_0 = \sum D_j \mu^j / j! \), \( \Delta = 2\pi \delta = \omega_0 - \omega_p \) is the pump detuning. The nonlinear coupling coefficient \( g \) (that is the Kerr frequency shift per photon) is defined as in \([3]\):

\[
g = \frac{\hbar \omega_0 c n_2}{n^2 V_0},
\]

where \( n \) and \( n_2 \) are the refractive and nonlinear optical indices, \( V_0 \) is the effective volume of the pumped mode, and \( c \) is the speed of light.

We then switch to dimensionless equations for convenience:

\[
i \frac{\partial \Psi}{\partial \tau} + \frac{1}{2} \frac{\partial^2 \Psi}{\partial \theta^2} + |\Psi|^2 \Psi - \zeta_0 \Psi = -i \Psi + if,
\]

\[
\Psi_{\text{out}} = \sqrt{\frac{5g\eta}{\kappa}} s_{\text{out}} = (f - 2\eta \Psi).
\]

\[\theta = \phi \sqrt{\frac{\kappa}{2D_2}}, \; f = \sqrt{\frac{5g\eta P_{\text{in}}}{\kappa\hbar \omega_0}}, \; \Psi = \sqrt{\frac{2\zeta_0}{\kappa}} A, \; \zeta_0 = 2\Delta / \kappa, \; \tau = \kappa t / 2\]

The left part of the equation (4) is the nonlinear Schrödinger equation (NLSE) for which the soliton solution is known

\[
S_0(\theta, \tau) = \sqrt{2(\zeta_0 + w)} \text{sech}(2\sqrt{\zeta_0 + w}\theta)e^{i\tau \tau + i\varphi_0}.
\]

The solution of (4) was also found \([4]\) in case of zero dissipation (the term \(-i \Psi \rightarrow 0\) as compared to \( \zeta_0 \Psi \)) which gives some insight on the behavior of the damped case. For the complete damped driven equation (4) only the flat (coordinate and time independent) solution is known:

\[
|C|^2 C - \zeta_0 C + iC = if,
\]

If the pump is strong enough \((f > (4/3)^{3/4} \approx 1.24)\) then the flat solution have three branches, two of which are stable and one is unstable. The detuning where the two stable states are possible are determined by the boundary values:

\[
\zeta_t \approx \frac{3}{2} (2f^2)^{1/3} - \frac{1}{2} (2f^2)^{-1/3} + O(f^{-2})
\]

\[
\zeta_h \approx f^2 + \frac{1}{4} f^{-2} + O(f^{-6})
\]
The lower boundary $\zeta_l$ is important for soliton switching with backward tuning under thermal nonlinearity described in the main paper. Transitions between multiple soliton states may occur near this effective detuning $\zeta_0$ ($\gamma = 1/\zeta_0 \ll 1$):

$$|C|^2 = C_0^2 = \frac{2}{3} \zeta_0 + \frac{2}{3} \sqrt{\zeta_0^2 - 3\cos \left( \frac{1}{3} \arccos \left( \frac{27f^2 - 2\xi_0^3 - 18\zeta_0}{2(\xi_0^3 - 3)^{3/2}} \right) + \frac{2}{3} \pi \right)}$$

$$C = C_0 e^{i\psi_0} = \frac{if}{|C|^2 - \zeta_0 + i} = -i\gamma f + \gamma^2 f + i\gamma^3 f - \gamma^4 (f + if^3) + O(\gamma^5),$$

$$C_0 = \gamma f - \frac{1}{2}\gamma^3 f + \gamma^4 f^3 + O(\gamma^5),$$

$$\psi_0 = -\frac{\pi}{2} + \gamma - \frac{1}{3}\gamma^3 + f^2 \gamma^4 + O(\gamma^5).$$

Equation (9)

However, it was also found in [5, 6] that for large enough detuning $\zeta_0$ a stationary soliton attractor may be well approximated as:

$$\Psi_0 = C + S_0(\theta),$$

$$S_0(\theta) = B_0 \text{sech}(B_0 \theta) e^{i\varphi_0}$$

Using the Lagrangian perturbation method [7] with this Ansatz it may be found that $B_0 = \sqrt{2\zeta_0}$ in agreement with (6) and $\varphi_0 = \arccos(\sqrt{8\zeta_0}/\pi f)$. From the last expression the upper boundary of soliton existence appears:

$$\zeta_0 < \frac{\pi^2 f^2}{8}$$

Equation (11)

Using the same Lagrangian approach we found next order approximation in terms of $1/\zeta_0$ using three parameters which provide better fit with numerical simulations for the smaller detuning $\zeta_0$, closer to lower boundary of existence.

$$\Psi = C + S(\theta),$$

$$S(\theta) = B_1 \text{sech}(B_2 \theta) e^{i\varphi_0}$$

$$B_1 = B_0 - \frac{5\pi}{8\zeta_0} f \cos(\chi)$$

$$B_2 = B_0 - \frac{\pi}{4\zeta_0} f \cos(\chi)$$

$$\chi = \varphi_0 - \psi_0 \approx \pi - \arcsin \left( \frac{\sqrt{8\zeta_0}}{\pi f} \right) + \frac{5\pi^2 - 64}{4\zeta_0 \pi^2} \approx \pi/2 + \varphi_0.$$ 

Equation (13)

Considering that separation between the solitons in a multiple soliton state is large enough to prevent their interaction, we may assume that multiple soliton state may be described as

$$\Psi_N = C + \sum_{k=1}^{N} S(\theta - \theta_j)$$

Equation (14)

Note that another solution for characteristic equation with $\chi \approx \arcsin \left( \sqrt{8\zeta_0}/\pi f \right)$ leads to unstable solitons also noted in [8]. The two solutions correspond to explicit soliton solutions $\psi_-$ and $\psi_+$ of undamped equation [4].

**SOLITON STEPS**

The unfiltered output signal on the detector is proportional to

$$U = \frac{1}{2\pi} \int |\Psi_{out}|^2 d\phi = |f - 2\eta C|^2 + N \sqrt{\frac{\kappa}{2D_2}} \frac{B_1}{B_2} \left( 8\eta^2 B_1 - 2\eta \pi(f - 2\eta C) e^{-i\varphi_0} + c.c. \right)$$

Equation (15)

Using approximations (10) one may calculate theoretical predictions for the soliton “steps”.

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NON-DESTRUCTIVE SOLITON PROBING

We add phase modulation to the pump (analysis for amplitude modulation is analogous) at normalized frequency \( \Omega = 4\pi \nu / \kappa \), where \( \nu \) corresponds to the experimental modulation frequency in GHz:

\[
f(t) = f_0 e^{i \epsilon \cos(\Omega t)} \approx f_0 [1 + \frac{i}{2} \epsilon e^{i \Omega t} + \frac{i}{2} \epsilon e^{-i \Omega t}],
\]

(16)

The signal on the VNA is determined in this way by a harmonic amplitude \( e^{U_{VNA}} \):

\[
U = \frac{1}{2\pi} \int |\Psi_{out}|^2 d\phi = U_0 + \epsilon U^- e^{-i \Omega t} + \epsilon U^+ e^{i \Omega t}
= U_0 + \epsilon U_{VNA} \cos(\Omega t + \phi_{VNA})
\]

(17)

By substitution of the flat solution \( C(t) \) into the equation (4) it is easy to find the background modulation (assuming that \( \epsilon \) is small):

\[
C(t) \rightarrow C + \epsilon e^{-i \Omega t} + \epsilon e^{i \Omega t}
\]

\[
e^\pm = \frac{f_0}{2} \frac{C^2 - 2|C|^2 + i \zeta_0 \pm \Omega}{\Omega^2 - \Omega^2 \pm 2i \Omega}
\]

\[
\Omega^2_\Sigma = (\zeta_0 - |C|^2)(\zeta_0 - 3|C|^2) + 1
\]

(18)

As the background \( |C| = C_0 \) is small, and \( \zeta_0 \gg 1, \Omega \zeta \approx \zeta_0 \), and hence \( \nu \zeta \approx \delta \).

For the simplest approximation of the first soliton VNA peak we may consider the modulation of the field as an additional perturbation of the Nonlinear Schrödinger Equation to find the solution response using Lagrangian technique with two or three parameter \((B_1 \neq B_2)\) soliton Ansatz.

\[
B_1(t) \approx B_1 + \epsilon b_1^- e^{-i \Omega t} + \epsilon b_1^+ e^{i \Omega t}
B_2(t) \approx B_2 + \epsilon b_2^- e^{-i \Omega t} + \epsilon b_2^+ e^{i \Omega t}
\varphi(t) \approx \varphi_0 + \epsilon \varphi^- e^{-i \Omega t} + \epsilon \varphi^+ e^{i \Omega t}
\]

(19)

This approach allows for a simple (in case of two-parametric Ansatz) solution which qualitatively well predicts the solitonic peak:

\[
\varphi^\pm = \frac{f \sqrt{2\zeta_0}}{2} \sin(\varphi_0)
\]

\[
\Omega^2_\Sigma = \pi f \sin(\varphi_0) \sqrt{2\zeta_0}
\]

\[
b_\Sigma^\pm = \frac{\pm i \Omega \varphi^\pm}{B_0}
\]

(20)

\( \Omega_\Sigma \) much weakly depends on detuning than \( \Omega_\Sigma \) and is closer to zero. The reason is that the soliton sidebands produced by nonlinear interaction of the soliton and modulated background have corresponding to pulse field distribution and “feel” the nonlinear detuning produced by the soliton while the flat background sidebands “feel” only the nonlinear shift produced by average intensity. The 3-dB-width of the resonances is the same (\( \kappa \) if transformed to unnormalized units) and corresponds to cold optical resonance width.

The above simplified derivation while allowing to obtain qualitative estimate, however, is not very accurate quantitatively as the soliton in fact interacts not directly with the pump but with the background. That is why we also performed analogous Lagrangian perturbation method for the modified NLSE obtained for \( S(\theta) \) function after substituting (12) in (4) with three parameter Ansatz. The solution is straightforward, though quite bulky and produces significantly better fit with numerical simulations.
NUMERICAL SIMULATION OF MODULATIONAL PROBING

To simulate the response of the soliton Kerr comb on a weak modulation we used the following system of coupled mode equations:

\[
\frac{\partial a_\mu}{\partial \tau} = -(1 + i\zeta_\mu - i\Theta) a_\mu + i \sum_{\mu' \leq \mu''} (2 - \delta_{\mu',\mu''}) a_{\mu'} a_{\mu''} a_{\mu'+\mu''-\mu} + \delta_{0\mu} f(\tau) \tag{21}
\]

Here \( \delta_{\mu',\mu''} \) is the Kronecker delta, \( a_\mu \) are normalized slowly varying amplitudes of the comb modes so that \( \Psi = \sum a_\mu e^{i\mu \phi} \). All mode numbers \( \mu \) are defined relative to the pumped mode \( \mu = m - m_0 \) with the initial azimuthal number \( m_0 \approx 2\pi R n_0/\lambda \), where \( \lambda = 2\pi c/\omega_p \) is the wavelength, \( R \) - is the microresonator radius. The pump phase modulation was taken as

\[
f(\tau) = f_0 e^{i\epsilon \sin(\Omega \tau)} \tag{22}
\]

The modulation amplitude \( \epsilon \) was taken small enough (\( \epsilon = 0.03 \) in the simulation below). As an initial condition at predefined detuning \( \zeta_0 \) and soliton number \( N \) theoretically calculated initial state was seeded using (14) with equally spaced \( N \) solitons. The equation was then propagated in time using adaptive Runge-Kutta. For convenience of postprocessing and to preserve high resolution for each frequency the simulation for different values of \( \Omega \) was done in one run changing \( \Omega \) in discrete steps for time duration of integer periods. For the same reason the modulation in (22) for smoother transition was chosen as sine and not cosine function. The result of the simulation for \( U = \sum |a_\mu|^2 \) is for \( N = 3 \), detuning \( \zeta_0 = 15 \) with 100 incremental steps for \( \Omega \) and 5 periods for each value is shown on Fig. 1 as well as extracted amplitude and phase VNA signals.

THERMALLY ENABLED SWITCHING OF SOLITON STATES

Our numerical model is based on the same system (21) of dimensionless coupled nonlinear mode equations modified to take into account the thermal effects. To this purpose an additional equation for the normalized variation of temperature \( \Theta = \frac{1}{n} \frac{dn}{dT} \frac{\delta T}{T} \) was solved simultaneously with conventional system. Throughout the simulations we neglected the frequency dependence of nonlinearity, losses and mode-overlap, interactions with other mode families, and any particularities of the resonator geometry. Thus, the modified set of coupled mode equations reads:
\[ \frac{\partial a_\mu}{\partial \tau} = -(1 + i \zeta_\mu(\tau - i \Theta))a_\mu + i \sum_{\mu' \leq \mu''} (2 - \delta_{\mu',\mu''})a_{\mu'}a_{\mu''}a_{\mu' + \mu'' - \mu} + \delta_{0,\mu}f \]

\[ \frac{\partial \Theta}{\partial \tau} = \frac{2}{\kappa \tau_T} \left( \frac{n_{2T}}{n_2} \sum |a_\mu|^2 - \Theta \right) \quad (23) \]

Here $\tau_T$ is the thermal relaxation time, $n_{2T}$ is the coefficient of thermal nonlinearity. For numerical analysis we consider the following parameters corresponding to Si$_3$N$_4$ microresonator: $\lambda = 1.553 \times 10^{-6}$ m, $n_0 = 2.4$, $n_2 = 2.4 \times 10^{-19}$ m$^2$/W, $V_0 = 10^{-15}$ m$^3$, $\kappa/2\pi = 3 \times 10^8$ Hz, $D_2/2\pi = 2.5 \times 10^6$ Hz, $D_3 = 0$, $\eta = 0.36$, $P = 2$ W. These parameters correspond to dimensionless force term $f = 5.8$. The ratio $n_{2T}/n_2 = 10$ was chosen to resemble experimental data, $\tau_T$ was chosen reasonably short to perform the simulation during a reasonable time.

In our simulations we used weak noise-like inputs and coupled-mode equations for 511 modes were numerically propagated in time using Runge-Kutta integrator. For analysis we extracted from numerical simulations the average intracavity intensity $\bar{U} = \sum |a_\mu|^2$. At first, slowly linearly increasing detuning we observed the characteristic step-like response pattern indicating soliton formation. The position of this “step” was shifted to larger values of the detuning due to the thermal effects as compared to purely Kerr case. The regime of multiple solitons inside the microresonator was usually observed, with different soliton numbers $N$ varied for different realizations of noise-like input. After the soliton regime was reached and stabilized, backward scan was initiated and the detuning was decreased with reduced velocity to provide thermal equilibrium. If the scan rate was sufficiently small, gradual decrease of soliton numbers on one per step was observed. However, at larger scan rates faster switches may be observed with missing single soliton steps and higher soliton number extinctions per step. It was found that at fixed pump power and thermal parameters the positions of the transitions between neighboring steps were fixed and did not vary for different inputs. We also checked that while in the absence of the thermal effects the positions of different steps are the same. We also confirmed that in equilibrium thermal effects may be taken into account from fast pure Kerr simulation by simple affine transform: $\bar{U} \rightarrow \bar{U}$, $\zeta_0 \rightarrow \zeta_0 + (n_{2T}/n_2)\bar{U}$.

**BACKWARD TUNING: ADDITIONAL EXPERIMENTAL DETAILS**

In this section we provide additional experimental details on the backward tuning of soliton states and the VNA measurements. As it was shown in the main paper, the power trace of the generated comb light, obtained in the forward (from shorter to longer wavelength) pump tuning reveals a thermal triangle with a step-like fall of power at the end. The appearance of the steps indicates soliton formation when the pump goes from the effectively blue-detuned operation regime to the effectively red-detuned regime. For the SiN samples studied in this work, typically, the fall consist of only one step that corresponds to the stochastically formed multiple soliton state. The unicity of the step implies that transition to multiple soliton states with lower number of solitons can be hardly achieved in the forward tuning. In order to track the statistics of the occurrences of different multiple soliton states, multiple forward scans over the resonance were recorded. Fig. 2 (a) shows the histograms of 200 overlaid oscilloscope traces of forward scans at different pump powers. The step distribution is discrete with respect to the generated light power and has equal power spacing between adjacent steps, indicating the formation of different number of solitons $N$. The intensity of the plot shows the probability to land on the corresponding soliton step. The steps distribution varies with pump power: the dominant multiple soliton state for a 2-W pump has $N = 8$ or $N = 9$, while for 3-W pump $N = 7$ or $N = 8$. At 4 W of the pump power the soliton steps are rarely observed. Fig. 1(b) shows the occurrences of different number of steps for various pump powers within the range of 2 – 4 W. It can be seen that the total number of traces showing soliton formation (dark blue, green, and yellow histogram bars) decreases as the pump power approaches 4 W.

In the main paper we showed the power trace of the generated comb light in the backward pump tuning from a multisoliton state which is represented by a staircase with equal steps (Fig.2 (g) of the main paper). The staircase repeats itself over multiple experimental runs, though the initial multiple soliton states and the tuning speed in the backward pump tuning can be different. Fig. 2(c) shows the two traces of backward tuning taken with delay of ca. 30 min and the same pump power of 3W. The tuning speed for the blue trace was 2.5 times slower than for the red trace. The both traces are appropriately matched in power levels as well as frequencies where the switching happens. The initial number of solitons for the faster trace was lower by 1. Also it experienced the switching in which two solitons were lost, however the overall staircase pattern was still maintained.

Fig. 2(d) shows the VNA measurements taken in the backward tuning process for two pump powers of 2.5 W and 3.6 W. The two-soliton switching for the higher power (from multiple soliton state with $N = 6$ to $N = 4$) was also
FIG. 2. **Additional experimental details on soliton switching.** (a) Histogram plots of 200 overlaid experimental traces (each) of the output comb light in the pump forward tuning over the resonance of 100 GHz Si$_3$N$_4$ microresonator at three pump powers of 2, 3 and 4 W with the same tuning speed. (b) Histogram of soliton steps occurrences in the forward pump tuning at different pump powers. The red colour corresponds to the absence of soliton states, the green color corresponds to the appearance of 1 soliton step, yellow - 2 steps, dark blue - more than 2 steps. For each pump power 200 traces were recorded. (c) Traces of the generated comb light in the froward tuning followed by backward tuning. The total time of the backward tuning for the blue trace is $\sim 250$ s, for the red trace is $\sim 100$ s. The pump power and the speed of the forward tuning were kept the same. (d) The power traces of the generated light obtained from 100 GHz Si$_3$N$_4$ microresonator for two pump powers of 2.5 and 3.6 W in the backward pump tuning from multiple soliton state with $N = 6$ (left) and $N = 7$ (right) to the effectively blue detuned regime. Sets of 500 concatenated VNA traces (each) that were taken during the backward tuning for each pump power are shown below the corresponding traces.

identified in the VNA map as a stronger separation between the C- and S-resonances in the VNA response after the switching.