SI 1. Form Factor Decomposition of CuO$_2$ IUC States

Here we present mathematical details behind the form factor organization of density waves on the CuO$_2$ plane. The different density wave form factors are due to periodic modulation of the $Q = 0$ form factors, whose point group symmetry is well defined. Modulating these form factors with wave-vector $Q \neq 0$, one obtains

$$A_S(r) = \begin{cases} S \cos(Q \cdot r + \phi_s), & r = r_{Cu}, \\ 0, & r = r_{Ox}, \\ 0, & r = r_{Oy}, \end{cases}$$

$$A_{S'}(r) = \begin{cases} S' \cos(Q \cdot r + \phi_{s'}), & r = r_{Cu}, \\ 0, & r = r_{Ox}, \\ S' \cos(Q \cdot r + \phi_{s'}), & r = r_{Oy}, \end{cases}$$

$$A_D(r) = \begin{cases} 0, & r = r_{Cu}, \\ D \cos(Q \cdot r + \phi_D), & r = r_{Ox}, \\ -D \cos(Q \cdot r + \phi_D), & r = r_{Oy}, \end{cases}$$

Equation S1.1 shows that a purely $d$-symmetry form factor density wave can be thought of as a wave on each of the oxygen sub-lattices but with a spatial phase shift of $\pi$ between them. The Fourier transform of the $d$-symmetry form factor density wave is presented in Figs. S2 and will be considered further in SI section 3.
SI 2: Setup Effect in SI-STM Measurements

Spectroscopic imaging scanning tunneling microscopy (SI-STM) measurements provide energy dependent electronic structure images that can be used to identify the presence of modulations whether they arise from dispersive quasi-particle interference (QPI) or ultra-slow-dispersive density waves (DW) – see K. Fujita et al *Strongly Correlated Systems - Experimental Techniques* by Springer (ISBN 978-3-662-44132-9). However, the protocol by which a tunneling junction is established in SI-STM measurements can transfer conductance modulations from one set of energies to another. This systematic error which results in a misidentification of the energy of states undergoing spatial modulations is called the setup effect. Therefore, determining the physically real modulations and especially their correct energy can present a grave challenge. Ignoring the setup effect, as is often the case, leads to incorrect characterization of electronic structure properties of materials.

I - Mathematical Description of STM Observables

The basic observable in STM experiments is the tunneling current, \( I \), which depends on the bias between the tip and the sample, \( V \), the tip sample separation, \( z \), and the position along the sample, \( r \):

\[
I(r, z, V) = f(r, z) \int_{0}^{V} LDOS(r, \epsilon) d\epsilon
\]  

(S2.1)

The function \( f(r, z) \) captures spatial variations due to surface corrugation, work function, matrix elements and proximity \( z \) of the tip to the surface. The integral of the local density of states, \( LDOS \), includes spatial variation of the electronic structure.

Spectroscopic imaging entails establishing a stable tunnel junction at every \( r \) by using the same arbitrary pre-chosen set points \( I_s \) and \( V_s \), and then measuring the
variation with $r$ of the current or of the differential conductance, at each bias $V$. The set point constraints then determine the pre-factor of the integral in equation (S2.1):

$$I(r,z,V_s) = I_s = f(r,z) \int_0^{eV_s} LDOS(r,\epsilon)d\epsilon$$

$$\Rightarrow f(r,z) = \frac{I_s}{\int_0^{eV_s} LDOS(r,\epsilon)d\epsilon}$$

and thus the expressions for the spectroscopic current and differential conductance $dI/dV$ are given by

$$I(r,V) \propto \int_0^{eV} LDOS(r,\epsilon)d\epsilon \frac{dI}{dV}(r,V) \propto \frac{LDOS(r,eV)}{\int_0^{eV_s} LDOS(r,\epsilon)d\epsilon}, \quad (S2.2)$$

The term in the denominator is responsible for the deleterious nature of the setup effect since it carries the imprint of electronic structure over the whole range of voltages between 0 and the set-up bias, $V_s$.

**II - Example of the Setup Effect**

The figure below demonstrates how the choice of setup bias to establish the tunneling junction strongly influences the acquired data. While both the left and right panels are spatial images of the differential conductance taken at $V = 50$ mV in the same field of view, the left was measured with $V_s = 150$ mV setup bias while the right with $V_s = 50$ mV. It is clear that the spatial intensity patterns of the same set of states in the material are imaged differently based on the setup bias parameter.
We emphasize that the situation shown in Fig. S1 is not an unusual effect but rather is completely typical and unavoidable in all SI-STM studies, except under some specially prepared conditions (Section 6).

SI 3. Predicted Fourier Transform STM Signatures of a dFF-DW

I - Sublattice Segregation Method to Determine DW Form Factor Symmetry

This section predicts the consequences of a primarily $d$-symmetry form factor density wave for the complex Fourier transform images of electronic structure on the three sublattices: $\text{Cu}(q)$, $\bar{\delta}_x(q)$, $\bar{\delta}_y(q)$. In turn this controls the three key functions used to measure modulations with different form factor symmetry as discussed in the main text.

\[
S(q) = \bar{\text{C}}u(q) \tag{S3.1}
\]
A purely $d$-symmetry form factor density wave has modulations in anti-phase on the x and y oxygen sub-lattices (Fig. 1B of main text) and no modulation on the Cu site. For the specific example of $Q_x \approx \left( \frac{1}{4}, 0 \right)$, $Q_y \approx \left( 0, \frac{1}{4} \right)$ considered in our study, this requires that the peaks at $\pm Q_x$ and $\pm Q_y$ present in both $\bar{\rho}_x(q)$ and $\bar{\rho}_y(q)$ must cancel exactly in $S'(q) = \left( \bar{\rho}_x(q) + \bar{\rho}_y(q) \right)/2$ and be enhanced in $\bar{D}(q) = \left( \bar{\rho}_x(q) - \bar{\rho}_y(q) \right)/2$. Conversely the peaks at $Q' = (1,0)Q_{xy}$ and $Q'' = (0,1)Q_{xy}$ will be enhanced in $S'(q)$ but will cancel exactly in $\bar{D}(q)$ (Figs S2B,C).

This occurs because the two sub-lattices have modulations at the same wave-vector but with a $\pi$ phase shift between them. Importantly, electronic structure images formed using the difference of oxygen sub-lattices, as in $\bar{D}(q)$, have the effect of removing this phase difference and recovering the peaks in the Fourier transform at the fundamental wave-vectors. These are necessary consequences of a density wave with a primarily $d$-symmetry form factor and hold for any $d$-symmetry form factor modulation in the presence of arbitrary amplitude and overall phase disorder.
Figure S2: Fourier Analysis of the Density Wave (A) Schematic of the segregated sublattice images $O_{xy}(r)$ and their Fourier transforms $\tilde{O}_{xy}(q)$. (B) $Re\tilde{S}'(q)$ and $Re\tilde{D}(q)$ for a d-form factor density wave with modulation along the x direction at $Q=(Q,0)$. Note that the origin of co-ordinates in real space has been chosen such that the Fourier transforms are purely real. (C) $Re\tilde{S}'(q)$ and $Re\tilde{D}(q)$ for a d-form factor density wave with modulations along the x and y directions at $Q=(Q,0),(0,Q)$. The key signature of the d-form factor is the absence of the peaks at $(Q,0),(0,Q)$ in $Re\tilde{S}'(q)$ and their presence in $Re\tilde{D}(q)$; the converse being true for the DW peaks surrounding $(\pm 1, 0) \text{ and } (0, \pm 1)$. © 2015 Macmillan Publishers Limited. All rights reserved
II - Form factor symmetry conversion of $q_5$ for $E > \Delta_0$

Fig. 3B of the main text shows the quasiparticle interference scattering vectors $q_1$-$q_7$ for energies below $\Delta_0$ which are primarily $s'$FF symmetry modulations in the $Z$-map. Above this energy scale the dFF modulation appears, plotted in blue at $Q^d_{x,y}$. This low-$q$ modulation is extracted according to the scheme presented above by which the sublattice subtraction procedure of $O_x$ and $O_y$ reveals the density wave at its primary wavevector. However, as seen from Fig. S2 B,C above, the physically identical dFF-DW can also be seen in the $\tilde{Q}_x(q) + \tilde{Q}_y(q)$ at a $q$-vector

$$Q^s_{x,y} = 1 - Q^d_{x,y}. \quad (S3.4)$$

Thus, the same physical modulation will manifest at different but related $q$-vectors phenomenon. In Fig. 3B, the dFF-DW peak in $\tilde{S}^s(q)$ is plotted as $Q^s_{x,y}$ while that in $\tilde{D}(q)$ is plotted as $Q^d_{x,y}$; they represent the same physical dFF-DW modulation and to within the systematic errors of the peak determination as shown, they are related by equation S3.4. In $\tilde{S}^s(q)$ the dFF-DW modulation appears as $Q^s_{x,y} \sim q_5(\Delta_0)$ persisting to energies much larger than $\Delta_0$. In $\tilde{D}(q)$ the identical physical modulation appears in the same energy range at a wavevector $Q^d_{x,y} \sim q_1(\Delta_0)$ (Fig. 3F).

![Figure S3: Equivalent representation of dFF modulations as $Q^s$ in the $S'$ and $Q^d$ in the $D$ channels.](image)
SI 4: Calculation of the local nematicity $F(r)$

We define the local nematicity of the DW to be the function

$$F(r) = \frac{|D_x(r)| - |D_y(r)|}{|D_x(r)| + |D_y(r)|}$$  \hfill (S4.1)

where $|D_x(r)|$ and $|D_y(r)|$ are the amplitudes of the $d$-symmetry form factor components of the density wave at wave-vectors $Q_x \approx \left(\frac{1}{4}, 0\right)$ and $Q_y \approx \left(0, \frac{1}{4}\right)$ respectively. Hence, in order to calculate this function we must first experimentally determine $|D_{x,y}(r)|$. This may be achieved through Fourier filtration of the measured $\bar{D}(q) = (\bar{D}_x(q) - \bar{D}_y(q))/2$ to select the amplitude contained in the region of reciprocal space proximately surrounding $Q_x$ and $Q_y$. The function $\bar{D}(q)$ is multiplied by a gaussian of FWHM $2\sqrt{2}\ln 2\Lambda$ centred on $Q_x$ and $Q_y$ respectively to create two filtered complex Fourier transforms $D_x(q)$ and $D_y(q)$.

$$\bar{D}_{x,y}(q) = 2\bar{D}(q)e^{-\frac{(q - Q_{x,y})^2}{2\Lambda^2}}$$  \hfill (S4.2)

Here, the factor of two arises from the fact that we have only filtered around $Q_{x,y}$ and not $-Q_{x,y}$ which, by the inversion symmetry of the Fourier transform of real functions, contain identical information. By taking the modulus of their inverse Fourier transforms one obtains an estimate of the amplitude of the x and y directed modulations.

$$|D_{x,y}(r)| = \left|\frac{1}{(2\pi)^2}\int dq e^{iq\cdot r}\bar{D}_{x,y}(q)\right|$$  \hfill (S4.3)
SI 5: Cuprate octet scattering QPI, calculation of energy resolved form factor magnitudes, and results for higher doped samples

I – The octet scattering vectors of Bogoliubov quasi-particle interference

Fig S4: Cuprate Octet Model for QPI Top panel shows the constant energy contours of the superconducting cuprate k-space electronic structure. The tips of the arc are connected via a set of 7 scattering vectors q₁-q₇. The bottom panel shows the dominant scattering vectors in q-space detected from quasiparticle interference imaging (QPI) in spectroscopic STM imaging. The scattering vectors can be used to reconstruct the energy resolved k-space electronic structure.
II – Spectroscopic determination of form factor magnitudes

The main texts presents the energy resolved weight of various form factor modulations in the spectroscopic data in Figure 3F. This section describes the set of operations to generate such curves.

First, sub-lattice segregation of the measured $Z(r,E)$ data is implemented by extracting the intensities at Cu, $O_x$ and $O_y$ sites from which, respectively, one constructs the energy resolved form factor images $S(r,E), S’(r,E)$, and $D(r,E)$, as prescribed in SI Section 3.

From these real space images one generates the power spectral density for each of the form factors, $|S^Z(q,E)|^2$, $|S’^Z(q,E)|^2$, and $|D^Z(q,E)|^2$. Fig. S3 presents a subset of these Fourier transform images taken at $E = 90\text{meV}$. Finally, integration is carried out over $q$-space regions inside the broken circles at each energy to obtain the spectral signature for each form factor, as plotted in Fig. 3F of the main text.

![Spectral Densities for Form Factors](image)

**Figure S5: Spectral Densities for Form Factors** Power spectral density for $S(A)$, $S(B)$, and $D(C)$ symmetry form factor at representative energy. Bragg peak locations are marked by “+”. $q$-space regions inside broken circles are the region of interest to calculate the spectral weight for each form factor.

SI 6: DW Phase Shift Between Empty and Filled States

Establishing the actual existence of a physical density wave at any given energy is the first step to determining the phase difference between modulations in the empty and filled states. In light of the setup effect, as introduced in SI Section 2, one must proceed carefully and not mistake the imprints of modulations in place of real ones.
To avoid such errors one must use a combination of spectroscopic and topographic imaging.

**I – Using Topographic Data to Determine Setup Bias to Avoid Setup Effect.**

In constant conductance imaging, or topographic imaging, the STM feedback system adjusts the tip sample separation, $z$, as it scans over the sample surface to maintain a set point current, $I_s$, at a constant applied tip-sample bias $V_s$. The topographic image, or $z(r)$, can be derived by starting with the equation for the tunneling current,

$$I(r,z,V) = f(r,z) \int_0^{eV} LDOS(r,\epsilon)d\epsilon$$  \hspace{1cm} (S6.1)

and assuming that the function $f(r,z)$, which represents the effect of corrugation, work function, and tunneling matrix elements, takes the form

$$f(r,z) = \exp(-\kappa z) \cdot A(r)$$  \hspace{1cm} (S6.2)

where $\kappa$ depends on a mixture of the work functions of the sample and tip. Then for topographic imaging the recorded value of the relative tip-sample separation takes the form

$$z(r) = \frac{1}{\kappa} \ln \left[ \int_0^{eV_s} LDOS(r,\epsilon)d\epsilon \right] + \frac{1}{\kappa} \ln \left[ \frac{I_s}{A(r)} \right]$$  \hspace{1cm} (S6.3)

The essential point is that a high signal-to-noise topographic image obtained by STM in constant conductance mode reveals contributions from both the surface structure and variations in the $LDOS(r,E)$ according to S6.3, obviously provided $E < eV_s$. In particular, through this effect a density wave imprints its signature logarithmically onto a topographic image beyond what would be expected from a slight deformation of the lattice due to electron-lattice coupling.
By choosing a setup bias such that the denominator in equation S2.2 is a constant without any periodic structure then any modulations observed in the differential tunneling conductance images, $g(r, E)$, must be representative of real modulations in the LDOS. As a matter of practicality, however, such a set up condition is highly unlikely and it is only possible to choose $V_s$ for which only a subset of modulations are not manifest. For the purposes of analyzing the cuprate density we have chosen $V_s$ so that the dFF-DW has no spectral weigh in the setup topographs and thus no weight in the denominator of S2.2 (see fig. S6 below).

From topography taken at different applied bias values (see equation S6.1), one can determine which bias produces a topographic image that has no dFF-DW signal arising from spatial modulations in the LDOS. The left panel of Fig. S6 presents a topographic image taken on UD-BSCCO at $V_s = 250\,\text{mV}$ while the right panel shows its corresponding $|\tilde{D}(\mathbf{q})|^2$, used to determine if there exists a dFF-DW component at $Q_{DW}$. There are no peaks at $Q_{DW}$ meaning that a spectroscopic setup bias of $V_s =$
250mV for measuring $g(r,E)$ will not lead to spurious imprints of dFF-DW modulation on the acquired $g(r,E)$ data. Hence, all dFF-DW modulations observed at any energy can be interpreted as physically real.

The main text presents $g(r,E)$ data for underdoped BSCCO. Because the data was acquired with the setup condition $V_s = 250$mV all dFF-DW modulations in all of the energy layers, both on the empty and filled sides, can be treated as physically real modulations and not due to the setup effect of Section 2. It is then meaningful to compare the phases of the dFF modulations between energy layers. With this justification Fig. 4E of main text presents the phase difference between $g(r,E)$ and $g(r,−E)$ without the setup effect error.

II Setup-Bias-Dependent DW Phase Shift

As described above, the correct choice of set up bias is essential in determining the existence of physically real density wave modulations on both the filled and empty states of the electronic structure. By choosing a bias for which the integrated density of states has no signature of the dFF-DW, it is ensured that there is no systematic error in the differential conductance data with a false dFF-DW signal. Such a requirement led us to search for and find a specific setup bias value $V_s > \Delta_1$ in order get a measure of the true phase shift between the empty and filled state density wave modulations (see SI Section V). However only a small range of such setup bias conditions with no topographic modulation at $Q$ plus which allowed for stable experimental conditions was achievable, meaning that repeating this procedure at several setup conditions proved impossible.

Nevertheless, to further demonstrate that the $\pi$ phase shift between empty/filled dFF-DW modulations cannot be due to the setup effect, we plot the value of this phase shift at $|E| = \Delta_1$ for a number of arbitrary $V_s$. If the phase difference were caused merely by the setup effect then the phase shift would always be $\pi$. But, as
the plot clearly demonstrates, the phase difference is dependent on the setup bias with the lower choices for $V_g$ producing arbitrary shifts due to the setup effect, as expected.

**Figure S7:** Phase shift between the empty and filled side density wave modulations at energy $\Delta_1$ for various spectroscopic setup bias values $V_g$. The data is shown for BSCCO-2212 UD45 for which $\Delta_1 \approx 80\text{meV}$.

**III– Phase Shift Analysis for Multiple Samples at Different Doping**

Figure S8 shows the phase difference between the dFF-DW modulation at $E=\pm\Delta_1$ in the differential conductance map for higher doped samples. As in Figs. 4E,F of the main text where the $|E| \sim \Delta_1$ density wave acquires a $\pi$ phase shift between empty and filled states for $p \sim 6\%$; here we show highly consistent effects for two other dopings so that the phenomenology actually spans the range $0.06 < p < 0.17$.

**FIG S8: Phase Shift in UD samples** dFF-DW phase shift between empty and filled for $p \sim 0.1$ and $p \sim 0.17$. 

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