Spectrum of controlling and observing complex networks

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I. Controllability Gramian

The complexity and challenges of controlling and observing complex systems arise from two sources: the underlying network representing the interactions between components (or nodes) and the intrinsic nonlinearity of nodal dynamics. Here we focus on linear dynamics that captures the behaviors of an nonlinear system around its equilibria. This way we naturally disentangle the impact of network structure from that of the dynamics nonlinearity. In Sec. VII we will extend the results to linearizable higher-order systems (e.g. epidemic spreading and power-network dynamics) and to nonlinear systems in the vicinity of their stable equilibria.

Formally, to control a linear time-invariant (LTI) system \( \dot{x}(t) = Ax(t) + Bu(t) \),

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
&= A x(t) + \sum_{j=1}^{N_D} B_{ij} u_j(t)
\end{align*}
\]

we aim for steering the system from an initial state to a desired final state by injecting appropriate control inputs \( u(t) = [u_1(t), u_2(t), \ldots, u_{N_D}(t)]^T \), where \( N_D \) is the number of driver nodes (or equivalently, the number of independent signals), the vector \( x(t) = [x_1(t), x_2(t), \ldots, x_{N_D}(t)]^T \) represents the states of the nodes at time \( t \), \( B \) is the input matrix with \( B_{ij} = 1 \) if control signal \( u_j(t) \) is imposed on node \( i \), and the adjacency matrix \( A \) captures the interactions between nodes, including the possibility of self-loops \( A_{ii} \) representing the self-regulation of node \( i \).

To control system (S1), the first and unavoidable step is to choose the driver nodes on which the control inputs \( u(t) \) need to be imposed [S1]. The recent advance offers an efficient algorithm to identify the driver nodes for arbitrary networks [S2, S3]. The second step is to design the appropriate inputs \( u(t) \) through which we can move the system from an initial state \( x_o \) to any desired final state \( x_d \) within time \( t \in [0, \tau] \). While there are an infinite number of possible inputs \( u(t) \) that can achieve the control, we focus on the optimal signal, aiming to minimize the required energy

\[
\int_0^\tau \|u(t)\|^2 \, dt
\]

that captures the energy for controlling electronic and mechanic systems or the amount of efforts to control.
I. CONTROLLABILITY GRAMIAN

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To control system (S1), the first and unavoidable step is to choose the driver nodes on which the control inputs \( u(t) \) need to be imposed [S1]. The recent advance offers an efficient algorithm to identify the driver nodes for arbitrary networks [S2, S3]. The second step is to design the appropriate inputs \( u(t) \) through which we can move the system from an initial state \( x_o \) to any desired final state \( x_d \) within time \( t \in [0, \tau] \). While there are an infinite number of possible inputs \( u(t) \) that can achieve the control, we focus on the optimal signal, aiming to minimize the required energy \( \int_0^\tau \|u(t)\|^2 dt \) that captures the energy for controlling electronic and mechanic systems or the amount of efforts to control.
biological and social systems [S4]. Without loss of generality, we assume that the system is in state \( x_o = 0 \) at \( t = 0 \). Based on the optimal control theory [S5], we have that the minimal energy required to move the system to point \( x_d \) in the state space is given by

\[
\mathcal{E}(\tau) = x_d^T G_c^{-1}(\tau) x_d,  \tag{S2}
\]

where \( G_c(\tau) = \int_0^\tau e^{A t} B B^T e^{A^T t} \, dt \) is the symmetric controllability Gramian [S5–S7]. When system (S1) is controllable, all eigenvalues of \( G_c(\tau) \) are real and positive. Therefore, the inverse of \( G_c(\tau) \)’s eigenvalues represent the energy required to move system (S1) through one unit distance in the corresponding eigen-directions.

For any stable network, the real parts of \( A \)'s eigenvalues are all negative. It has been shown that the control energy \( \mathcal{E}(\tau) \) converges to a steady value when the time \( \tau \) increases [S6]. Henceforth we focus on the control energy \( \mathcal{E} \equiv \mathcal{E}(\tau \to \infty) \) and the controllability Gramian \( G \equiv G_c(\tau \to \infty) \). In the following we derive the controllability Gramian \( G \) for stable undirected networks. Since in this case matrix \( A \) can be decomposed as \( A = V A V^T \), where \( V \) represents the eigenvectors, \( \Lambda = \text{diag}\{-\lambda_1, -\lambda_2, \ldots, -\lambda_N\} \) are the eigenvalues and \( \lambda_i \) represents the absolute eigenvalue of \( A \), i.e., \( \lambda_i > 0 \) for all \( i \), the controllability Gramian of LTI system (S1) can be cast to

\[
G = \int_0^\infty e^{A t} B B^T e^{A^T t} \, dt \\
= V \left[ \int_0^\infty e^{A t} (V^T B B^T V) \, e^{A^T t} \, dt \right] V^T \\
= V \left[ \int_0^\infty (V^T B B^T V) \circ S(t) \, dt \right] V^T  \tag{S3}
\]

where \( \circ \) is the Hadamard product (i.e., \( Z = X \circ Y \) means \( Z_{ij} = X_{ij} Y_{ij} \)), and the time-
varying matrix $S(t) = \{S_{ij}(t)\} \in \mathbb{R}^{N \times N}$ with $S_{ij}(t) = e^{-(\lambda_i + \lambda_j)t}$. Hence

$$G = V \left[ (V^TBB^T) \circ \left\{ \int_0^\infty e^{-(\lambda_i + \lambda_j)t} \, dt \right\} \right] V^T$$

$$= V \left[ (V^TBB^T) \circ \left\{ \frac{1}{\lambda_i + \lambda_j} \right\} \right] V^T$$

$$= V \left[ (V^TBB^T) \circ C \right] V^T,$$  
(S4)

where the matrix $C = \{C_{ij}\} \in \mathbb{R}^{N \times N}$ with $C_{ij} = \frac{1}{\lambda_i + \lambda_j} > 0$. That is how we obtained Eq. (3) in the main text.

II. Network Eigenvalues

A. Eigenvalue distribution

For stable undirected networks, we consider the self-loops, i.e., the diagonal elements of matrix $A$ as $A_{ii} = -\delta - \sum_{j=1}^N A_{ij}$. When $\delta = 0$, $A$ becomes the Laplacian matrix of the network and this kind of self-loops has been widely used to model several dynamical processes taking place on networks, such as opinion dynamics [S8] and synchronization of oscillators [S9, S10]. For convenience, here $\delta$ is a small positive number within the interval $(0, 1]$, ensuring that the network is stable, i.e. all eigenvalues of $A$ are negative [S6].

If $\delta = 0$, according to [S11], for scale-free networks with power-law degree distribution $p(k) \sim k^{-\gamma}$, the distribution of the $(N - 1)$ nonzero absolute eigenvalues follows a power law with the same exponent $-\gamma$, i.e., $p(\lambda) \sim \lambda^{-\gamma}$. If $\delta > 0$, obviously, each absolute eigenvalue of $A$ increases by $\delta$. The distribution of the corresponding absolute eigenvalues is also a power law with the same exponent $-\gamma$ (see Fig. S1a). For weighted networks, we randomly draw the link weights from the uniform distribution in $(0, 1]$. In this case, we find that the distribution of the absolute eigenvalues also displays a power-law tail with the same exponent $-\gamma$ (see Fig. S1a).
FIG. S1: a, The distribution of A’s absolute eigenvalues. The network is generated using the static model [S12], with the network size $N = 4000$, the average degree $\langle k \rangle = 8$, and the degree distribution $p(k) \sim k^{-\gamma}$ where $\gamma = 2.5$. The self-loops, i.e. the diagonal elements of matrix A, are $A_{ii} = -\delta - \sum_{j=1}^{N} A_{ij}$ with $\delta = 0.2$. For the weighted network, the weights of links are drawn randomly from the uniform distribution within $(0,1]$. b, Each absolute eigenvalue $\lambda_r$ as a function of its rank $r$. $r = 1$ and $r = N$ correspond to the largest and the smallest eigenvalues respectively. According to our analytical prediction (S8), the slope is $-1/(\gamma - 1) = -2/3$, agreeing well with numerical results for both unweighted and weighted scale-free networks. Note that the cut-off for high-rank eigenvalues is induced by the small-eigenvalue saturation of the distributions in a.

B. Eigenvalue gaps

We sort the absolute eigenvalues of the stable matrix $A$ in ascending order $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_N$, allowing us to derive the gaps between any two nearest eigenvalues. Consider $\lambda_i$ as a function of $i$. Since the index $i$ is uniformly distributed in the interval $[1, N]$, the probability density $p(i) = 1/N$. According to probability theory we have $p(\lambda)d\lambda = p(i)di$. When the distribution of the absolute eigenvalues follows $p(\lambda) \sim \lambda^{-\gamma}$,
we have
\[ \frac{d\lambda}{di} = \frac{p(i)}{p(\lambda)} \sim \frac{\lambda^\gamma}{N}. \] (S5)

Thus, \( \int_{\lambda}^{\lambda_{\text{max}}} \lambda^{-\gamma}d\lambda \sim \frac{1}{N} \int_{i}^{N} di \), leading to
\[ \lambda \sim \left( \frac{N}{a-i} \right)^{\frac{1}{\gamma+1}}, \] (S6)

where \( a \) is a constant that depends on \( \lambda_{\text{max}} \), i.e., the boundary condition. To determine the value of \( a \), we recall that for the distribution \( p(\lambda) \sim \lambda^{-\gamma} \) the maximum eigenvalue is \( \lambda_{\text{max}} \sim N^{\frac{1}{\gamma+1}} \) (derived from the constraint \( \int_{\lambda_{\text{max}}}^{\infty} p(\lambda)d\lambda \sim \frac{1}{N} \) [S13]). Therefore, when \( i = N \), \( \lambda_{\text{max}} \equiv \lambda_N \sim N^{\frac{1}{\gamma+1}} \). Plugging this boundary condition into (S6), we obtain
\[ a = N + 1 \] and hence
\[ \lambda_i \sim \left( \frac{N}{N+1-i} \right)^{\frac{1}{\gamma+1}}, \] (S7)

as stated in the main text. To test our theoretical prediction, we consider each absolute eigenvalue \( \lambda_r \) as a function of its rank \( r \) so that \( r = 1 \) or \( r = N \) corresponds to the largest or smallest absolute eigenvalue, respectively. Obviously, from (S7) we obtain
\[ \lambda_r \sim \left( \frac{N}{r} \right)^{\frac{1}{\gamma+1}} = \left( \frac{r}{N} \right)^{-\frac{1}{\gamma+1}}. \] (S8)

As shown in Fig. S1b, for both unweighted and weighted scale-free networks our theoretical prediction (S8) agrees well with numerical results.

III. CONTROL ENERGY FOR \( N_D = 1 \)

A. Derivation

If all nodes exhibit nonidentical self-loops, we can control an undirected network by driving a single node [S14]. Mathematically, an undirected network is controllable by driving a single node if and only if \( A \)'s eigenvalues are distinct from each other [S3, S15]. In
this case $V^TBB^TV = \{V_{ih}V_{jh}\} \sim \mathcal{O}(1/N)$, where $h$ is the index of the driver node. Thus, $V^TBB^TV$ can be viewed as a small perturbation to the matrix $\{C_{ij}\} = \{\frac{1}{\lambda_i+\lambda_j}\}$ in (S4), i.e. the statistical behavior of $G$’s eigenvalues is mainly determined by the eigenvalues of $C$. Based on [S16] $C$’s eigenvalues can be approximated by its Cholesky factors $\{\varrho_i\}$. According to (S7), if $\gamma \to 0$, i.e., for extremely heterogeneous networks [S17–S19], the eigenvalue gaps $g_i \equiv \lambda_{i+1} - \lambda_i$ are identical. For $\gamma \to \infty$ (homogeneous networks), $g_i$ becomes uniform also. Hence, to gain the insights into the heterogeneity of the eigen-energies required to move the system in different directions, we reasonably assume that $g_i = g$ for all $i$, resulting in

$$g_i = \frac{1}{2\lambda_i} \prod_{j=1}^{i-1} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\right)^2 = \frac{1}{2ig} \left[i!(i-1)!\right]^2,$$  

(S9)

To derive the distribution of eigen-energies, we recast (S9) into

$$\ln g_i = -\ln(2ig) + 2 \ln(i!) + \ln((i-1)!) - \ln((2i-1)!)).$$

Using the Stirling’s formula $\ln(i!) \approx i \ln(i) - i$ for large $i$, we arrive at

$$\ln g_i \approx -\ln(2ig) + 2 \left(i \ln \left(\frac{i(i-1)}{(2i-1)^2}\right) + \ln \left(\frac{2i-1}{i-1}\right)\right)$$
$$\approx -\ln(2ig) - 2(2i \ln 2 - \ln 2)$$
$$= -[\ln(2ig) + (4i - 2) \ln 2].$$  

(S10)

As the eigen-energies are the inverse of Controllability Gramian $G$’s eigenvalues, based on (S10) we have $E_i \approx 1/g_i$, i.e.

$$E_i \approx 2ige^{(2i-1)\ln 4}.$$  

(S11)
Similar to Sec. I, we consider the index \( i \) as a continuous variable, thus

\[
p(\mathcal{E}) = p(i)\left|\frac{di}{d\mathcal{E}}\right|.
\]

From (S11) we obtain that \( \ln \mathcal{E}_i \approx \ln(2i\gamma) + (2i - 1)\ln 4 \approx 2i\ln 4 \), hence

\[
i \approx \frac{\ln \mathcal{E}_i}{2\ln 4}, \tag{S13}
\]

and

\[
\frac{d\mathcal{E}_i}{di} \sim 2g \left( e^{(2i-1)\ln 4} + 2ie^{(2i-1)\ln 4} \ln 4 \right)
= \mathcal{E}_i(2\ln 4 + 1/i). \tag{S14}
\]

Inserting (S13) and (S14) into (S12),

\[
p(\mathcal{E}) \sim \frac{1}{2\mathcal{E}(1 + 1/\mathcal{E})\ln 4}. \tag{S15}
\]

Finally, for large \( \mathcal{E} \), we obtain the distribution of the eigen-energies

\[
p(\mathcal{E}) \sim \mathcal{E}^{-1}, \tag{S16}
\]

as shown in the main text (Eq. (5)).

**B. Validation**

To obtain the eigen-energy distribution (S16) from (S9), we have used the approximation of Stirling’s formula and treated the discrete index \( i \) as a continuous variable. To test the accuracy of our approximation, we consider the complementary cumulative distribution (CCD): \( p_>(\mathcal{E}) \equiv \int_{\mathcal{E}}^{\mathcal{E}_{\max}} p(\mathcal{E})d\mathcal{E} \). When \( p(\mathcal{E}) \sim \mathcal{E}^{-1} \) as in Eq. (S16), we have that \( p_>(\mathcal{E}) \sim (\ln \mathcal{E}_{\max} - \ln \mathcal{E}) \), decreasing linearly with \( \ln \mathcal{E} \). Fig. S2 shows the distri-
FIG. S2: Validation of the prediction (S16). The complementary cumulative distribution (CCD) $p(\varepsilon)$ is numerically calculated from (S9) with $\varepsilon_i = 1/g_i$. Indeed, $p(\varepsilon)$ decreases linearly with $\log \varepsilon$, indicating that $p(\varepsilon) \sim \varepsilon^{-1}$.

Distribution $p(\varepsilon)$ numerically calculated from Eq. (S9) (in this paper we use Advanpix, a high-precision computing toolbox for MATLAB to do the numerical calculations [S20]). We find that, indeed, $p(\varepsilon)$ decreases linearly with $\ln \varepsilon$, agreeing very well with the theoretical prediction (S16).

C. Maximum control energy

Maximum control energy represents the required energy to move system S1 in the most difficult direction. To derive the maximum control energy $\varepsilon_{\text{max}}$ (i.e. $\varepsilon_N$) for the distribution (S16), we use the same technique as in Sec. II: $p(\varepsilon)d\varepsilon = p(i)di$, where $i$ is the index and $p(i)$ is constant. Therefore, $d\varepsilon/\varepsilon \sim di$, leading to

$$\varepsilon_{\text{max}} \equiv \varepsilon_N \sim e^N.$$ (S17)
IV. CONTROL ENERGY FOR $1 < N_D < N$

In the main text we derive that, if $N_D = N$ the eigen-energy distribution $p(\mathcal{E}) \sim \mathcal{E}^{-\gamma}$ (Eq. (4) and Figs. 2a, b), where $\gamma > 2$ is the exponent of the network’s power-law degree distribution; if $N_D = 1$ the eigen-energy distribution $p(\mathcal{E}) \sim \mathcal{E}^{-1}$ (Eq. (5) and Figs. 2c, d), almost independently of the network structure. To understand the transition from $p(\mathcal{E}) \sim \mathcal{E}^{-\gamma}$ for $N_D = N$ to $p(\mathcal{E}) \sim \mathcal{E}^{-1}$ for $N_D = 1$, here we show the numerical results for the in-between cases, i.e. when $1 < N_D < N$.

We consider the distribution $p(\hat{\mathcal{E}})$, where $\hat{\mathcal{E}} \equiv \ln \mathcal{E}$. Obviously, if $p(\mathcal{E}) \sim \mathcal{E}^{-\gamma}$, $p(\hat{\mathcal{E}}) \sim e^{(1-\gamma)\hat{\mathcal{E}}}$. When $N_D = N$, $p(\hat{\mathcal{E}})$ is exponential for $\gamma > 2$ and has only one peak (see Fig. S3a); When $N_D = 1$, $p(\mathcal{E}) \sim \mathcal{E}^{-1}$ and $p(\hat{\mathcal{E}})$ becomes uniform (see Fig. S3e); When $1 < N_D < N$, $p(\hat{\mathcal{E}})$ has multi peaks (see Figs. S3b, c, d). The number of peaks $N_{\text{peak}} = \text{int}[N/N_D]$, where $\text{int}[x]$ is the rounding-up integer of $x$. For example, $N_{\text{peak}} = 1, 2, 4, 5$ for $N_D/N = 1.0, 0.6, 0.3, 0.2$ respectively (see Fig. S3).

Now we look in-depth at the multi-peak nature of $p(\hat{\mathcal{E}})$. Note that each peak in $p(\hat{\mathcal{E}})$ corresponds to a band of eigen-energies as shown in Fig. 3b of the main text. We index the energy bands by $\alpha$ and denote the boundary of the $\alpha$-th band by $\mathcal{E}_\alpha$, i.e. $\mathcal{E}_\alpha$ is the eigen-energy corresponding to the boundary of the $\alpha$-th energy band. As shown in Fig. S4, we find that, approximately, $\ln \mathcal{E}_\alpha \sim \alpha$. When $\alpha = N/N_D$, $\mathcal{E}_\alpha$ becomes the boundary of the last eigen-energy band, thus the maximum control energy $\mathcal{E}_\text{max} \sim e^\alpha = e^{N/N_D}$.

V. DIRECTED NETWORKS

For general directed weighted networks, $A_{ij}$ and $A_{ji}$ can be unequal to each other when $i \neq j$. There are two cases: (i) $A$ is not diagonalizable, i.e. there does not exist any invertible matrix $V$ such that $V^{-1}AV$ is a diagonal matrix; or (ii) $A$ is diagonalizable, i.e. there exists an invertible matrix $V$ such that $A = V\Lambda V^{-1}$ where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Note that the elements of eigenvectors $V$ and the eigenvalues in diagonal matrix $\Lambda$ can be complex rather than real as for undirected networks. Although we still lack analytical
Note that, when the system is controllable, all eigenvalues of $N$ energies for controlling the network in the eigen-directions of forward neural networks [S22]. In this case, one can numerically calculate correspond to directed acyclic networks, such as food-webs, citation networks, or feed-forward neural networks [S22]. In this case, one can numerically calculate $e^{At}$ in (S3) as $e^{At} = \sum_{l=0}^{\infty} \frac{1}{l!} (At)^l = \sum_{l=0}^{m} \frac{1}{l!} (At)^l$.

(i) If $A$ is not diagonalizable, most notably a nilpotent matrix [S21], i.e. there exists a finite integer $m$ so that $A^l = 0$ when $l > m$. Note that nilpotent matrices correspond to directed acyclic networks, such as food-webs, citation networks, or feed-forward neural networks [S22]. In this case, one can numerically calculate $e^{At}$ in (S3) as $e^{At} = \sum_{l=0}^{\infty} \frac{1}{l!} (At)^l = \sum_{l=0}^{m} \frac{1}{l!} (At)^l$.

(ii) If $A$ is diagonalizable but has complex eigenvectors and eigenvalues, we can numerically address the issues of control energy and observational uncertainty as follows. We decompose matrix $A$ as $A = V\Lambda V^{-1}$ where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. For stable systems, $\text{Re}\lambda_i < 0$ for all $i$. Thus the controllability Gramian in (S2) becomes

$$G = V \left[ \left( (V^{-1}BB^T(V^{-1})^T) \circ C \right) \right] V^T, \quad (S18)$$

where $C_{ij} = \frac{1}{\lambda_i + \lambda_j}$. $G$ is symmetric and the inverse of its eigenvalues represent the eigenenergies for controlling the network in the eigen-directions of $N$-dimensional state space. Note that, when the system is controllable, all eigenvalues of $G$ are real and positive.
Note that, when the system is controllable, all eigenvalues of $G$ are real and positive. 

In this case, one can numerically calculate $\gamma = 3$, which mists a finite integer in this paper.

Here we show that these issues can be addressed numerically using the framework proposed tools for the control energy and observational uncertainty in general directed networks, 

$\mathbb{A}$

As decomposing matrix $\mathbb{A}$ is diagonalizable but has complex eigenvectors and eigenvalues, we can numerically address the issues of control energy and observational uncertainty as follows. We decompose matrix $\mathbb{A}$ as

$$
\mathbb{A} = \mathbb{V} \mathbb{\Lambda} \mathbb{V}^{-1}
$$

where $\mathbb{\Lambda} = \text{diag} \{ \lambda_1, \lambda_2, ..., \lambda_N \}$, $\lambda_i$ is the inverse of its eigenvalues represent the eigenvalues of $\mathbb{A}$.

Thus the controllability Gramian in (S2) becomes

$$
\langle \mathbb{G} \rangle = \sum_{i,j} l_{ij} \mathbb{G}_{ij}
$$

FIG. S3: The distributions $p[\log(\epsilon)]$ for different fraction of driver nodes $\frac{N_D}{N}$. The network is generated using the static model [S12] with degree distribution $p(k) \sim k^{-\gamma}$ with $\gamma = 3$. The number of nodes $N = 100$ and the average degree $\langle k \rangle = 10$. In the Erdős-Rényi (ER) network, $N = 100$ and $\langle k \rangle = 14$. The eigen-energy bands are indexed by $\alpha$, and $\ln \mathcal{E}_\alpha$ denotes the boundary of the $\alpha$-th band.

FIG. S4: The boundaries of eigen-energy bands. The scale-free (SF) network, generated by the static model [S12], has degree distribution $p(k) \sim k^{-\gamma}$ with $\gamma = 3$. The number of nodes $N = 100$ and the average degree $\langle k \rangle = 10$. In the Erdős-Rényi (ER) network, $N = 100$ and $\langle k \rangle = 14$. The eigen-energy bands are indexed by $\alpha$, and $\ln \mathcal{E}_\alpha$ denotes the boundary of the $\alpha$-th band.

FIG. S5: Control spectrum of a directed network. Network size $N = 50$ and the average in-degree and out-degree are all 3. With the maximum matching method[S1, S2] we find that the minimum number of driver nodes for controlling this network is 3. The left panel shows the control spectrum for different number of driver nodes $N_D$. The right panel shows the maximum control energy $\log(\mathcal{E}_{\text{max}})$ as a function of $N/N_D$. 

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Here we take a 50-node directed network for an example to demonstrate the applicability of the numerical method described above. We numerically calculate the controllability Gramian $G$ according to (S18) and then the control spectrum are simply the inverse of $G$’s eigenvalues. We plot the results for different number of driver nodes in Fig. S5, showing that for this directed network the maximum control energy (corresponding to the most difficult direction in the state space) increases exponentially when the number of driver nodes decreases.
VI. REAL NETWORKS

A. Datasets and basic characteristics

Table S1: Real networks analyzed in the paper. For each network, we show its type, name and reference, number of nodes $N$ and edges $L$, clustering coefficient $c$, assortativity (degree-degree correlation) $r$, maximum modularity $Q$, and number of dead ends $N_{de}$.

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<td>north-euro-grid[S28]</td>
<td>236</td>
<td>320</td>
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<td>-0.11</td>
<td>0.79</td>
<td>31</td>
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<td>AS-Dec1998[S24]</td>
<td>493</td>
<td>1,145</td>
<td>0.18</td>
<td>-0.24</td>
<td>0.48</td>
<td>127</td>
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<td>1,504,957</td>
<td>0.84</td>
<td>-0.32</td>
<td>0.16</td>
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<td>-0.24</td>
<td>0.48</td>
<td>0</td>
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<td>0.27</td>
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<td>18,625</td>
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<td>0.38</td>
<td>0.42</td>
<td>0</td>
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<td></td>
<td>gene-coexpress[S38]**</td>
<td>139</td>
<td>431</td>
<td>0.35</td>
<td>0.25</td>
<td>0.59</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>mutualism-net[S39]</td>
<td>135</td>
<td>2,928</td>
<td>0.84</td>
<td>-0.32</td>
<td>0.16</td>
<td>1</td>
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<td></td>
<td>cat-cortex[S40]***</td>
<td>95</td>
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<td>-0.09</td>
<td>0.29</td>
<td>0</td>
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<td>453</td>
<td>2,025</td>
<td>0.65</td>
<td>-0.23</td>
<td>0.44</td>
<td>6</td>
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FIG. S6: Control spectrum for randomized real networks (\(N_D = N\)). Each panel shows the distribution \(p(\mathcal{E})\) of eigen-energies for controlling the real network and the degree-preserved randomized network. The prediction \(p(\mathcal{E}) \sim \mathcal{E}^{-\gamma}\) is shown as straight lines, where \(\gamma\) is the exponent of the power-law degree distribution for each network. The error bars represent standard deviations.

* The network is composed of the airlines among top 150 airports in United States.
** The edge weights of this network are very heterogeneous so that its largest eigenvalue is larger than \(N\).
*** This is a connected component in the human gene-coexpression network, corresponding to mitochondrial metabolism and redox homeostasis.
**** This is the symmetrized [S42] cat cortex network containing all cortical and thalamic areas.

To calculate the maximum modularity we use the algorithm proposed in [S43]. For real weighted networks we treat the node strength (the sum of the node’s link weights) as the node’s effective degree in the prediction of eigen-energy distributions.

### B. Beyond the degree distribution

As shown in Table S1 real networks have local clustering coefficient, degree-degree correlations and community structure. To assess the impact of these topological characteristics on the theoretical predictions, we perform the degree-preserved randomization
FIG. S7: Control spectrum for randomized real networks ($N_D = 1$). Each panel shows the complementary cumulative distribution $p>\mathcal{E}$ of eigen-energies for controlling the real network and the degree-preserved randomized network. The prediction $p(\mathcal{E}) \sim \mathcal{E}^{-1}$ is shown as straight lines, which is a first-order approximation of the eigen-energy distributions and agrees well with the numerical results. The reason for the deviations between analytical and numerical results is that for these networks the eigenvalue gaps of $A$ are not exactly equal.
FIG. S8: $\varepsilon_{\text{max}}$ vs $N_D$ for randomized real networks. Our prediction $\varepsilon_{\text{max}} \sim e^{N/N_D}$ is shown as dash lines. The error bars represent standard deviations.
[S44] to each network, eliminating the local clustering, degree correlations and modularity.

Fig. S6 shows the distribution $p(\mathcal{E})$ of eigen-energies required to control the randomized networks when $N_D = N$. We find that indeed $p(\mathcal{E}) \sim \mathcal{E}^{-\gamma}$ as predicted by Eq. (4) of the main text, where $\gamma$ is the power law exponent of the corresponding degree (or effective degree) distribution. Fig. S7 shows the complementary cumulative distribution $p_>(\mathcal{E})$ for $N_D = 1$, indicating that for randomized networks the eigen-energies follow the distribution $p(\mathcal{E}) \sim \mathcal{E}^{-1}$ as predicted by Eq. (5) of the main text. Fig. S8 shows the maximum control energy $\mathcal{E}_{\text{max}}$ for moving the randomized network towards its most difficult direction, agreeing excellently with the prediction $\mathcal{E}_{\text{max}} \sim e^{N/N_D}$.

As shown in Table S1 many real networks have dead ends, i.e. nodes with one degree. It has been demonstrated that dead ends can undermine the stability of complex systems [S28]. Here, to test the impact of dead ends on control energy we employ several real networks that contain a considerable number of dead ends, including open-flights, AS-CAIDA, power grid (US), north-euro-grid, AS-Dec1998, inter-companies, PPI (human), hetero-net-human and gene-coexpress. As shown in Figs. 2-4 and S6-S8 our findings about the maximum control energy and the distribution of eigen-energies are robust against dead ends.

VII. LINEAR DYNAMICS AND BEYOND

Our controllability spectra framework is pertinent directly to several linear systems and as well to a variety of higher-order systems that can be cast in an exact linear form [S45]. It also offers the first-order approximation to a wide class of nonlinear systems when we aim to control them locally.
A. Linear systems

Opinion dynamics and consensus - A simple yet widely-used model for opinion dynamics reads as [S8, S49]

\[ \dot{x}_i = -\sum_{j \in \mathcal{N}_i} a_{ij} (x_i - x_j), \]  

(S19)

which can be obviously recast into formalism (1) by adding external control inputs. Note that the time-varying form of (S19) has found broad applications in vehicle formations, coordinated decision making, and distributed algorithm design [S50].

B. Linearizable high-order systems

Epidemic spreading - Epidemic spreading [S48] is an important dynamical process taking place on networks. Here we consider the Susceptible-Infected-Susceptible model

\[ \dot{x}_i = -\alpha_i x_i + (1 - x_i) \beta_i \sum_{j \in \mathcal{N}_i} a_{ij} x_j, \]  

(S20)

which has been adopted to describe the transmission of infectious disease in human populations, the spreading of virus in computer networks, and complex contagions in social networks. In (S20) \( x_i \) captures the probability of node \( i \) being infected/informed, \( \alpha_i \) and \( \beta_i \) captures the recovery/forgetting and infection rates of node \( i \) respectively. It is well known that when the infection rate is below or around a critical threshold, the virus can not diffuse widely in the network, i.e. \( x_i \ll 1 \) [S48]. In this situation, we can apply external control to the network system through

\[ \dot{x}(t) = Ax(t) + Bu(t), \]  

(S21)

where \( A_{ij} = \beta_i a_{ij} \) and \( A_{ii} = -\alpha_i \).
Power network - The dynamics of a power network with $N$ generators can be described by the swing equation [S46]

$$m_i \ddot{x}_i + d_i \dot{x}_i = - \sum_{j \in N_i} w_{ij} (x_i - x_j),$$  \hspace{1cm} (S22)

where $x_i$ is the phase angle of generator $i$, $m_i$ and $d_i$ are the inertia and damping coefficients respectively. The edge weight $w_{ij}$ represents the susceptance of the power line connecting generators $i$ and $j$, $N_i$ denotes the set of $i$’s neighbor nodes.

We rewrite the generators’ phase angles as $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$. Let $\mathbf{x}_1 \equiv x$ and $\mathbf{x}_2 \equiv \dot{x}$, (S22) can be cast into

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ L M^{-1} & -D M^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$  \hspace{1cm} (S23)

where $I$ is the identity matrix, $L_{ij} = w_{ij}$ for $i \neq j$ and $L_{ii} = -\sum_{j=1}^{N} w_{ij}$, $D = \text{diag}\{d_1, d_2, \ldots, d_N\}$, and $M = \text{diag}\{m_1, m_2, \ldots, m_N\}$. Denoting $\mathbf{z} \equiv (\mathbf{x}_1 \mathbf{x}_2)$ and $A \equiv \begin{pmatrix} 0 & I \\ L M^{-1} & -D M^{-1} \end{pmatrix}$, we can impose external control to the system through

$$\dot{\mathbf{z}}(t) = A \mathbf{z}(t) + B \mathbf{u}(t),$$  \hspace{1cm} (S24)

where $A$ represents the new network constructed by (S23) (see also [S47]), $B$ is the input matrix and $\mathbf{u}(t)$ is the control inputs. Thus, our results presented in the paper can be easily extended to high-order systems.
C. Local control of nonlinear systems

In the following we discuss the required energy for local control of nonlinear systems whose dynamics have the general form

\[ \dot{x}(t) = f(x, u, t), \]  

(S25)

where \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \) describes the states of the \( N \) nodes at time \( t \), \( f(*) = [f_1(*), f_2(*), \ldots, f_N(*)]^T \) captures the system dynamics, and \( u(t) = [u_1(t), u_2(t), \ldots, u_{N_D}(t)]^T \) represents the \( N_D \) external control inputs.

Linearizing the dynamics (S25) around its stable fixed point \( x^* \) gives

\[ \dot{x}(t) = f(x^*, u^*, t) + \frac{\partial f}{\partial x}|_{x^*, u^*} (x - x^*) + \frac{\partial f}{\partial u}|_{x^*, u^*} (u - u^*). \]  

(S26)

Since \( x^* \) is a fixed point, we have \( f(x^*, u^*, t) = 0 \). To ensure the stability of the fixed point the Jacobian matrix \( A \equiv \frac{\partial f}{\partial x}|_{x^*, u^*} \) needs to be negative definite, i.e. all of its eigenvalues are negative. Matrix \( B \equiv \frac{\partial f}{\partial u}|_{x^*, u^*} \) is the input matrix correspondingly. Thus, (S26) can be rewritten as

\[ \dot{x}(t) = A(x - x^*) + B(u - u^*). \]  

(S27)

Let \( z(t) = x(t) - x^* \) and \( v(t) = u(t) - u^* \), we arrive at

\[ \dot{z}(t) = Az(t) + Bv(t), \]  

(S28)

exactly the same form as Eq. (1) in the main text, indicating that Eq. (1) indeed captures the controlled dynamics of a nonlinear system around its stable fixed point.

When the linearized system (S28) is controllable, the original nonlinear system (S25) is locally controllable [S51, S52]. For local control, i.e. fine tuning the system in the vicinity around \( x^* \), one may want to constrain the system’s trajectory \( x(t) \) to be close enough to \( x^* \) [S7, S51].
To do so we can impose a penalty to \( ||\mathbf{x}(t) - \mathbf{x}^*|| \), i.e. \( ||\mathbf{z}|| \), aiming at minimizing the objective function [S5]

\[
J = \int_0^T (\mathbf{z}^T Q \mathbf{z} + \mathbf{v}^T \mathbf{v}) dt,
\]

where \( T \) is the given control time and \( Q = \text{diag}\{q, q, \ldots, q\} \) with \( q \geq 0 \). Note that if \( q = 0 \) (S29) reduces to the minimum-energy control as discussed in the main text. It is fairly expected that when \( q \) increases the resulting trajectory \( \mathbf{x}(t) \) of the controlled system becomes closer to \( \mathbf{x}^* \).

To minimize (S29) for system (S28), i.e. \( \dot{\mathbf{z}} = A \mathbf{z} + B \mathbf{v} \) with the boundary condition of initial state \( \mathbf{z}_o = \mathbf{0} \) and desired final state \( \mathbf{z}(T) = \mathbf{z}_d \), we defining the Hamiltonian [S5]

\[
H(t) = \frac{1}{2} (\mathbf{z}^T Q \mathbf{z} + \mathbf{v}^T \mathbf{v}) + \lambda^T (A \mathbf{z} + B \mathbf{v}).
\]  

(S30)

According to the Pontryagin’s Maximum Principle [S53] the optimal input \( \mathbf{v}(t) \) and the corresponding trajectory \( \mathbf{z}(t) \) obey the following equations [S5]

\[
\begin{cases}
\begin{pmatrix}
\dot{\mathbf{z}} \\
\dot{\lambda}
\end{pmatrix} =
\begin{pmatrix}
A & -BB^T \\
-Q & -A^T
\end{pmatrix}
\begin{pmatrix}
\mathbf{z} \\
\lambda
\end{pmatrix}, \\
\mathbf{v}(t) = -B^T \lambda(t).
\end{cases}
\]  

(S31)

Next we derive the optimal control input \( \mathbf{v}(t) \) with regard to the objective (S29) and the corresponding trajectory \( \mathbf{z}(t) \) of the controlled system. Denoting \( M = \begin{pmatrix} A & -BB^T \\ -Q & -A^T \end{pmatrix} \),
we have

\[
\begin{pmatrix}
\mathbf{z} \\
\lambda
\end{pmatrix} = e^{Mt}
\begin{pmatrix}
\mathbf{z}_o \\
\lambda_o
\end{pmatrix}.
\]  

(S32)

For undirected and weighted networks \( A \) is symmetric, thus the eigenvalues of \( M \) are all real. We decompose \( M = V \Lambda V^{-1} \) where \( V \) is composed of \( M \)'s eigenvectors and \( \Lambda \) is the diagonal matrix containing the corresponding eigenvalues. From (S32) one can see that the state \( \mathbf{z}(t) \) depends on \( \lambda_o \), providing a way to determine \( \lambda_o \) given the boundary
conditions \( z(T) = z_d \) and \( z_0 = 0 \). For convenience we denote \( V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \), \( V^{-1} = \begin{pmatrix} (V^{-1})_{11} & (V^{-1})_{12} \\ (V^{-1})_{21} & (V^{-1})_{22} \end{pmatrix} \) and \( \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \), arriving at

\[
\lambda_0 = (V_{11} e^{\Lambda_1 T} (V^{-1})_{12} + V_{12} e^{\Lambda_2 T} (V^{-1})_{22})^{-1} z_d. \tag{S33}
\]

Thus, the trajectory of the controlled system is

\[
z(t) = (V_{11} e^{\Lambda_1 t} (V^{-1})_{12} + V_{12} e^{\Lambda_2 t} (V^{-1})_{22})(V_{11} e^{\Lambda_1 T} (V^{-1})_{12} + V_{12} e^{\Lambda_2 T} (V^{-1})_{22})^{-1} z_d. \tag{S34}
\]

and the evolution of \( \lambda \) is

\[
\lambda(t) = (V_{21} e^{\Lambda_1 t} (V^{-1})_{12} + V_{22} e^{\Lambda_2 t} (V^{-1})_{22})(V_{11} e^{\Lambda_1 T} (V^{-1})_{12} + V_{12} e^{\Lambda_2 T} (V^{-1})_{22})^{-1} z_d. \tag{S35}
\]

Inserting (S35) into (S31) we obtain the control energy

\[
\mathcal{E}(T) \equiv \int_0^T v(t)^T v(t) dt = z_d^T W z_d. \tag{S36}
\]
where

\[ W = \left( (V^{-1})_{12}^T e^{A_1^T V_{11}^T} + (V^{-1})_{22}^T e^{A_2^T V_{12}^T} \right)^{-1} \]

\[ \left( \int_0^T ((V^{-1})_{12}^T e^{A_1^t V_{21}^T} + (V^{-1})_{22}^T e^{A_2^t V_{22}^T}) BB^T (V_{21} e^{A_1^t (V^{-1}_{12})} + V_{22} e^{A_2^t (V^{-1}_{22})}) dt \right) \]

\[ (V_{11} e^{A_1^t (V^{-1}_{12})} + V_{12} e^{A_2^t (V^{-1}_{22})})^{-1}. \]  \hspace{1cm} (S37)

Eq.(S36) shows that the control energy depends on desired final state \( z_d \), indicating the variability of control energy for different directions in the vicinity of the stable fixed point \( x^* \). We illustrate the result using the same three-node network as in Fig. 1 of the main text. Fig. S9a shows that when \( q \) increases the trajectory indeed becomes more local around \( x^* \). Yet, the control energy for different directions are highly heterogeneous no matter how large or small is \( q \), implying that if we aim to locally control a nonlinear system around its stable fixed point there still exist certain directions that are energetically prohibitive.

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S34. Centeno, M. A. International phone traffic, country to country flows (accessed in March 2015);  http://www.princeton.edu/~ina/telephone/itu_phonedata.xls


