Topological boundary modes in isostatic lattices

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SUPPLEMENTARY ONLINE MATERIALS

Proof of Index Theorem

In this appendix we provide details of the proof of the index theorem discussed in the text. Our starting point is equation (6), which describes the zero-mode count in a region \( S \) of a larger system. Using the fact that \( \{ H, \tau^z \} = [\rho_S(\hat{r}), \tau^z] = 0 \), it is straightforward to check that equations (6-8) imply that

\[
\nu_T^S = \frac{1}{2} \lim_{\epsilon \to 0} \text{Tr} \left[ \tau^z \frac{1}{\mathcal{H} + i\epsilon} [\rho^S(\hat{r}), \mathcal{H}] \right].
\]

Since \([\rho^S, \mathcal{H}] = 0\) for \( \rho^S = 1 \), and \( \mathcal{H} \) has a finite range \( a \), \( \nu_T^S \) comes only from the boundary of region \( S \) from where \( \rho^S(r) \) varies. If we assume that the boundary region is gapped and \( \rho(\hat{r}) \) varies slowly on the scale \( L \gg a \), then we safely take \( \epsilon \) to zero and expand to leading order in \( a/L \). Since \([\rho^S(\hat{r}), \mathcal{H}]_{\alpha\beta} = \mathcal{H}_{\alpha\beta}(\rho^S(r_\alpha) - \rho^S(r_\beta)) \sim \mathcal{H}_{\alpha\beta}(r_\alpha - r_\beta) \cdot \nabla \rho(\hat{r}) \), we may write

\[

\nu_T^S = \frac{1}{2} \text{Tr} \left[ \tau^z \nabla \rho(\hat{r}) \cdot \mathcal{H}^{-1}[\hat{r}, \mathcal{H}] \right].

\]

We next suppose that in the boundary region the lattice is periodic, so that the trace may be evaluated in a basis of plane waves:

\[
|k, a\rangle = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}} \exp[i \mathbf{k} \cdot (\mathbf{R} + \mathbf{d}_a)] |\mathbf{R} + \mathbf{d}_a\rangle,
\]

where \( \mathbf{R} \) is a Bravais lattice vector in a system with periodic boundary conditions and \( N \) unit cells. \( \mathbf{d}_a \) are basis vectors for the \( d_n + n_b \) sites and bonds per unit cell. The phases are chosen such that the position operator is \( \hat{r} \sim i \nabla_k \). In this basis, the Bloch Hamiltonian \( \mathcal{H}_b(k) \) is a \( d_n + n_b \times \) square matrix with off diagonal blocks \( Q(k) \) and \( Q^*(k) \), where

\[
Q_{ab}(k) = (k, a|Q|k, b).
\]

\( \nu_T^S \) then has the form

\[
\nu_T^S = \int_{\partial S} d^{d-1} S \mathbf{P}_T \cdot \hat{n}
\]

where the integral is over the boundary of \( S \) with inward normal \( \hat{n} \), and

\[
\mathbf{P}_T = \int_{BZ} d^d k \left( \frac{2\pi}{d} \right)^d \text{Im Tr}[Q^{-1} \nabla_k Q].
\]

It is useful, to write

\[
\text{Im Tr}[Q^{-1} \nabla_k Q] = \nabla_k \text{Im log det } Q.
\]

FIG. 1: Evaluating the zero mode count. a) Cylindrical geometry for evaluating the zero mode count for a domain wall between \( \mathbf{R}_T^1 \) and \( \mathbf{R}_T^2 \), indicated by the dashed line. b) Cylindrical geometry for evaluating the zero mode count for a surface indexed by reciprocal lattice vector \( \mathbf{G} \). The region \( S \) covers half the cylinder. The boundary \( \partial S \) is deep in the interior. b) also shows our notation for the surface unit cell.

It is then straightforward to show that

\[
\det Q(k + \mathbf{G}) = \det Q(k) \exp[-i \mathbf{G} \cdot \mathbf{r}_0].
\]

where

\[
\mathbf{r}_0 = d \sum_{\text{sites } i} \mathbf{d}_i - \sum_{\text{bonds } m} \mathbf{d}_m.
\]

For a general lattice, \( \mathbf{r}_0 \) is non-zero. However, if the coordination number of site \( i \) is \( z_i \), then \( \mathbf{r}_0 = \sum_i (d - z_i/2) \mathbf{d}_i + \mathbf{R} \)

where \( \mathbf{R} \) is a Bravais lattice vector. Thus, for an isostatic lattice with uniform coordination \( z = 2d \), \( \mathbf{r}_0 \) is a Bravais lattice vector, and it is always possible to shift \( \mathbf{d}_m \) by lattice vectors to make \( \mathbf{r}_0 = 0 \). In the text of the paper, we assumed \( \mathbf{r}_0 = 0 \).

Here we will keep it general, and show that while \( \mathbf{r}_0 \) affects \( \nu_T^S \), its effect is canceled by a compensating term in \( \nu_T^S \).

For the general case, let us write \( \text{det } Q(k) = q_0(k) \exp[-i \mathbf{k} \cdot \mathbf{r}_0] \), where \( q_0(k) = q_0(k + \mathbf{G}) \) is periodic in the BZ. Equation (6) then involves two pieces:

\[
\mathbf{P}_T = \frac{1}{V_{\text{cell}}} [-\mathbf{r}_0 + \mathbf{R}_T].
\]

Here \( \mathbf{R}_T \) is a Bravais lattice vector describing the winding numbers of the phase of \( q_0(k) \) around the cycles of the BZ. It may be written \( \mathbf{R}_T = \sum_i n_i \mathbf{a}_i \), with

\[
n_i = \frac{1}{2\pi i} \int_{C_i} \text{d}k \cdot \nabla_k \log q_0(k)
\]

where as in the text, we assume that for a given cycle \( C_i \) of the BZ the winding number is path independent.

Application to zero modes at a domain wall

To determine the zero mode count at a domain wall between topological states \( \mathbf{R}_T^1 \) and \( \mathbf{R}_T^2 \), we consider a cylinder perpendicular to the domain wall (or a similar construction for \( d \) dimensions). We expect the zero mode count to be proportional to the “area” \( A \) (or length in 2D) of the domain wall. We will, therefore, be interested in the zero mode count per unit cell, \( \nu^S/N_{\text{cell}} \), where \( N_{\text{cell}} = A/A_{\text{cell}} \), and \( A_{\text{cell}} = V_{\text{cell}}/d_{\text{cell}} \) is the projected area of the surface.
unit cell, which can be expressed in terms of the volume of the bulk unit cell $V_{\text{cell}}$ and the distance $d_{\text{cell}}$ between Bragg planes. Referring to Supplementary Fig. 1a, we use equation (10) to evaluate equation (5) away from the domain wall to give $\nu_T = (A/V_{\text{cell}})(R_T^1 - R_T^2) \cdot \hat{n}$, where $\hat{n}$ is the unit vector pointing to the right. The zero mode count per unit cell can be expressed in terms of the reciprocal lattice vector $G = 2\pi \hat{n}/d_{\text{cell}}$ that indexes the domain wall as

$$\nu_T^S / N_{\text{cell}} = G \cdot (R_T^1 - R_T^2) / 2\pi.$$  

(12)

*Application to zero modes at the edge*

We next determine the number of zero modes localized on a surface (or edge in 2d) indexed by a reciprocal lattice vector $G$. Consider a cylinder with axis perpendicular to $G$ and define the region $S$ to be the points nearest to one end of the cylinder, as shown in Supplementary Fig. 1b. A similar construction can be used to count the zero modes on a surface in $d$ dimensions.

$\nu_T^S$ is determined by evaluating equation (??) deep in the bulk of the cylinder where the lattice is periodic. From equation (6) we may write

$$\nu_T^S / N_{\text{cell}} = G \cdot (R_T - r_0) / 2\pi.$$  

(13)

The local count, $\nu_L^S$, depends on the details of the termination at the surface and is given by the macroscopic “surface charge” that arises when positive charges $+d$ are placed on the sites and negative charges $-1$ are placed on the bonds. As discussed in the text, it can be determined by evaluating the dipole moment of a unit cell with site and bond vectors $\tilde{d}_a$ that is defined so that the surface can be accommodated with no left over sites or bonds. This unit cell is in general different from the unit cell used to compute $\nu_T^S$, and its dipole moment is in general not quantized. However, since the difference is due to a redefinition of which bond is associated with which unit cell, the dipole moment differs from $r_0$ by a Bravais lattice vector,

$$R_L = d \sum_{\text{sites } i} \tilde{d}_i - \sum_{\text{bonds } m} \tilde{d}_m - r_0.$$  

(14)

It follows that the local count may be written

$$\nu_L^S / N_{\text{cell}} = G \cdot (R_L + r_0) / 2\pi.$$  

(15)

The total zero mode count on the edge is then

$$\nu^S / N_{\text{cell}} = G \cdot (R_L + R_T) / 2\pi.$$  

(16)

It can be seen that the dependence on $r_0$, which depends on the arbitrary unit cell used to define $\nu_T^S$, cancels.