

Supplemental Material to “Spasers, VCSELs, and Surface plasmon emitting diodes (SPED’s): their unique features and key performance metrics”.

Supplement 1

In this supplement we explain the basics of using energy balance to obtain the fractions of energy contained in different forms. We start with the model for classical Lorentzian **oscillator**, because dispersion of most materials in consideration can be approximated as combination of several Lorentzian oscillators (Sellmeyer’s formulas) or an integral over it. The relative dielectric permittivity ε_r and susceptibility $\chi(\omega)$ of such materials can be written as

$$\varepsilon_r(\omega) = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}. \quad (1)$$

We do not consider imaginary part of susceptibility here for simplicity, but one can show that it does not significantly change any of the conclusions obtained here. Consider the electro-magnetic field $E \cos \omega t$, its time-averaged energy density of the electrical energy is

$$\bar{U}_E = \frac{1}{4} \frac{\partial(\omega \varepsilon_r)}{\partial \omega} \varepsilon_0 E^2 = \frac{1}{4} \varepsilon_0 E^2 \left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} + \frac{2\omega_p^2 \omega^2}{(\omega_0^2 - \omega^2)^2} \right] = \frac{1}{4} \varepsilon_g \varepsilon_0 E^2 \quad (2)$$

where ε_g is the group dielectric constant. Note that the last term in the square brackets, $\omega \partial \chi / \partial \omega$ becomes prominent only when ω approaches ω_0 and thus can be called the “dispersive” component of the group dielectric constant, $\chi_d = 2\omega_p^2 \omega^2 / (\omega_0^2 - \omega^2)^2$. Thus we can write

$$\varepsilon_g(\omega) = 1 + \chi(\omega) + \chi_d(\omega). \quad (3)$$

For a standard cavity with a standing wave it follows from the Maxwell equation $\nabla \times \mathbf{H} = -\varepsilon_0 \varepsilon_r (\partial / \partial t) \mathbf{E}$ that the magnetic field is $H \sin \omega t = \varepsilon_0^{1/2} \varepsilon_r^{1/2} \mu_0^{-1/2} E \sin \omega t$ and the time-dependent magnetic energy density is

$$U_M(t) = \frac{1}{2} \mu_0 H^2 \sin^2 \omega t = \frac{1}{2} \varepsilon_0 \varepsilon_r E^2 \sin^2 \omega t \quad (4)$$

and the time average value of it is obviously $\bar{U}_M = \frac{1}{4} \varepsilon_0 \varepsilon_r E^2$.

Next we shall determine the time dependence of the electrical energy. Physically, there are three parts to it. First of all there is a “vacuum” contribution

$$U_V(t) = \frac{1}{2} \varepsilon_0 E^2 \cos^2 \omega t. \quad (5)$$

Second, there is a contribution due to the potential energy of the oscillating dipoles, proportional to the square of the amplitude of bound electrons oscillations, which can be easily determined from the equation of motion for the Lorentzian oscillator as

$$U_p(t) = \frac{1}{2} \varepsilon_0 \frac{\omega_p^2 \omega_0^2}{(\omega_0^2 - \omega^2)^2} E^2 \cos^2 \omega t. \quad (6)$$

Third, there is a contribution due to the kinetic energy of the oscillating dipoles, proportional to the square of the velocity of bound electron oscillations,

$$U_K(t) = \frac{1}{2} \varepsilon_0 \frac{\omega_p^2 \omega^2}{(\omega_0^2 - \omega^2)^2} E^2 \sin^2 \omega t = \frac{1}{2} \varepsilon_0 \frac{1}{2} \chi_d E^2 \sin^2 \omega t. \quad (7)$$

If we take time average of the components (5) -(7) and add them up, it is easy to see that we obtain precisely the expression (2), which explains physical reason for use group dielectric constant to estimate energy. But one can also transform (6) into

$$U_p(t) = \frac{1}{2} \varepsilon_0 \left[\frac{\omega_p^2 \omega_0^2}{\omega_0^2 - \omega^2} + \frac{\omega_p^2 \omega^2}{(\omega_0^2 - \omega^2)^2} \right] E^2 \cos^2 \omega t = \frac{1}{2} \varepsilon_0 \left[\chi(\omega) + \frac{1}{2} \chi_d(\omega) \right] E^2 \cos^2 \omega t. \quad (8)$$

We can now write the expression for total energy density of the mode as

$$\begin{aligned} U(t) &= U_V(t) + U_p(t) + U_K(t) + U_M(t) = \\ &= \frac{1}{2} \varepsilon_0 E^2 \left[(1 + \chi) \cos^2 \omega t + \frac{1}{2} \chi_d \cos^2 \omega t + \frac{1}{2} \chi_d \sin^2 \omega t + \varepsilon_r \sin^2 \omega t \right] = \frac{1}{2} \varepsilon_0 E^2 \left[\varepsilon_r + \frac{1}{2} \chi_d \right] = \bar{U} \end{aligned} \quad (9)$$

As one can see there exists a perfect balance as the broadly defined “capacitive (potential)” energy $U_C = U_V + U_p$ gets transferred into the broadly defined “inductive (kinetic)” energy (sum of actual kinetic energy of dipoles and magnetic energy) $U_L = U_M + U_K$. This energy balance,

$$U_V + U_p = U_M + U_K \quad (10)$$

which can be written precisely as what enables the oscillations in the first place. We can now estimate the average energy contained in the fields,

$$\bar{U}_F = \bar{U}_V + \bar{U}_M = \frac{1}{4} \varepsilon_0 E^2 (1 + \varepsilon_r), \quad (11)$$

and the average energy contained in the oscillation of electrons in the dipoles.

$$\bar{U}_E = \bar{U}_p + \bar{U}_K = \frac{1}{4} \varepsilon_0 E^2 (\chi + \chi_d). \quad (12)$$

The fraction of energy contained in the field is then

$$f_{field} = \frac{1 + \varepsilon_r}{2\varepsilon_r + \chi_d} = \frac{1 + \varepsilon_r}{\varepsilon_r + \varepsilon_g} \quad (13)$$

which is of course equation (2) in the main text.

Next we consider a metal with dielectric constant

$$\varepsilon_{r,m}(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \quad (14)$$

The electric field energy is now

$$\bar{U}_{EM} = \frac{1}{4} \frac{\partial(\omega \varepsilon_r)}{\partial \omega} \varepsilon_0 E^2 = \frac{1}{4} \varepsilon_0 E^2 \left[1 + \frac{\omega_p^2}{\omega^2} \right] = \frac{1}{4} (1 + \chi_{km}) \varepsilon_0 E^2. \quad (15)$$

It is easy to see that the potential energy of free electrons is zero and the second term corresponds to the kinetic energy of the electrons. The total electrical energy of field in the metal is

$$U_{E,m}(t) = U_V(t) + U_{K,m} = \frac{1}{2} \epsilon_0 E^2 \cos^2 \omega t + \frac{1}{2} \epsilon_0 E^2 \chi_{km} \sin^2 \omega t. \quad (16)$$

This energy balance makes perfect sense as it shows that self-sustaining oscillations in the metal can take place only when “kinetic susceptibility” $\chi_{km} = 1$, i.e. at plasma frequency.

Now let us consider the mode that is confined to the sub-wavelength dimension $a \ll \lambda / n$ where n is the refractive index. Using the same Maxwell equation one can obtain the relation between electric and magnetic field amplitudes $H \approx \zeta \epsilon_0^{1/2} \epsilon_r^{1/2} \mu_0^{-1/2} E$, where $\zeta = \pi n a / \lambda$ is the measure of how small relative to the wavelength the mode is. Of course when $\zeta \ll 1$ the magnetic field becomes very small which is the well-known static limit. We now see that with the magnetic energy being only

$$U_M(t) = \frac{1}{2} \zeta^2 \epsilon_0 \epsilon_r E^2 \sin^2 \omega t, \quad (17)$$

the energy balance (10) cannot be maintained and the energy will get radiated (**this is of course nothing but an alternative derivation of the diffraction limit!**). But if the metal is present inside the cavity, the kinetic energy of electrons will contribute to the inductive (“kinetic”) energy and restore the balance as

$$U_V + U_P = U_M + U_K + U_{K,m}. \quad (18)$$

Of course for each particular geometry the energy balance can be restored only at a certain discrete set of frequencies which are, of course, the eigen-frequencies of the SP modes. The fraction of energy inside the field now becomes

$$f_{\text{field}} = \frac{1 + \zeta^2 \epsilon_r}{2 \epsilon_r + \chi_d} = \frac{1 + \zeta^2 \epsilon_r}{\epsilon_r + \epsilon_g} \quad (19)$$

which is exactly Equation (3) of the main text. Note that using energy balance between “capacitive (potential)” energy and “inductive (kinetic)” we can determine the fraction of energy stored in the field even if we do not know the precise form of the metal dispersion!

Supplement 2

Here we derive equations (4) of the main text and evaluate the Purcell factor, gain, spontaneous emission time and other parameters. As we have discussed, most of these parameters do not directly influence the threshold current of a spaser, but are important in their own right.

According to the Fermi Golden rule, the rate of stimulated transition between two states in conduction and valence bands with energies E_c and E_v separated by the energy $\hbar\omega$ is

$$R(\mathbf{r}) = \frac{2\pi}{\hbar} \frac{1}{3} \frac{e^2 P^2}{m_0^2 \omega^2} (f_c - f_v) \left| \frac{\mathbf{E}_\omega(\mathbf{r})}{2} \right|^2 \delta(E_c - E_v - \hbar\omega) \quad (20)$$

where \mathbf{E}_ω is the electric field of the polariton mode with frequency ω , P is the Kane's matrix element of momentum, and f_c and f_v are the Fermi functions of two bands, defined as

$$f_{c,v}(E) = \frac{1}{e^{\frac{E-\mu_{c,v}}{kT}} + 1} \quad (21)$$

where $\mu_{c,v}$ are the chemical potentials of two bands that are related to the electron-hole pair density \tilde{n} as

$$\tilde{n}(\mathbf{r}) = \frac{1}{2\pi^2} \left(\frac{2m_{c,v}^*}{\hbar^2} \right)^{3/2} \int E_{c,v}^{1/2} \frac{1}{e^{\frac{E_{c,v}-\mu_{c,v}}{kT}} + 1} dE_{c,v}. \quad (22)$$

Next invoking the relation between the electric field strength and energy density in the mode, we obtain

$$U_\omega(\mathbf{r}) = \frac{1}{4} \varepsilon_0 [\varepsilon_g(\mathbf{r}) + \varepsilon_m(\mathbf{r})] |E_\omega(\mathbf{r})|^2 \quad (23)$$

where

$$\varepsilon_m(\mathbf{r}) = \frac{\mu_0 |\mathbf{H}(\mathbf{r})|^2}{\varepsilon_0 |\mathbf{E}(\mathbf{r})|^2} \quad (24)$$

is the relative fraction of magnetic energy, $\varepsilon_m(\mathbf{r}) \ll 1$ for a substantial sub-wavelength cavity, and $\varepsilon_m(\mathbf{r}) \approx \varepsilon_r$ for the conventional waveguide modes. Then, integrating (20) over the joint density of states

$$\rho_{c,v} = \frac{1}{2\pi^2} \left(\frac{2\mu_r}{\hbar^2} \right)^{3/2} (\hbar\omega - E_g)^{1/2} \quad (25)$$

where E_g is the bandgap energy and

$$\mu_r = \frac{1}{m_c^{-1} + m_v^{-1}} \quad (26)$$

is the reduced effective mass, one arrives at the rate of the stimulated decay of the carrier density $\tilde{n}(\mathbf{r})$

$$\frac{d\tilde{n}(\mathbf{r})}{dt} = \frac{1}{[\varepsilon_g(\mathbf{r}) + \varepsilon_m(\mathbf{r})]} \frac{1}{3\pi\hbar\varepsilon_0} \frac{e^2 P^2}{m_0^2 \omega^2} \left(\frac{2\mu_r}{\hbar^2} \right)^{3/2} (\hbar\omega - E_g)^{1/2} (f_c - f_v) U_\omega(\mathbf{r}). \quad (27)$$

Next we can calculate the total energy of n_p polaritons in the mode as

$$\hbar\omega n_p = \int U_\omega(\mathbf{r}) dV = U_{\omega 0} V_{eff} \quad (28)$$

where $U_{\omega 0}$ is the peak energy density, usually occurring at the metal surface and

$$V_{eff} = \frac{U_{\omega 0}}{\int U_\omega(\mathbf{r}) dV} \quad (29)$$

is the effective mode volume. In our numerous prior works we have shown that for the subwavelength mode the effective volume has the same order of magnitude as the volume of nanoparticle itself. Finally we normalize the energy density to $U_{\omega 0}$ as

$u_{\omega}(\mathbf{r}) = U_{\omega}(\mathbf{r}) / U_{\omega 0}$, so that $\int u_{\omega}(\mathbf{r}) dV = V_{\text{eff}}$ and obtain from (27)

$$\begin{aligned} -\frac{d\tilde{n}(\mathbf{r})_{\text{stim},\omega}}{dt} &= \frac{1}{[\varepsilon_g(\mathbf{r}) + \varepsilon_m(\mathbf{r})] V_{\text{eff}}} \frac{1}{3\pi\omega\varepsilon_0} \frac{e^2 P^2}{m_0^2} \left(\frac{2\mu_r}{\hbar^2}\right)^{3/2} (\hbar\omega - E_g)^{1/2} (f_c - f_v) u_{\omega}(\mathbf{r}) n_p = \\ &= \frac{R_{\text{stim}}(\mathbf{r}) u_{\omega}(\mathbf{r})}{V_{\text{eff}}} n_p \end{aligned} \quad (30)$$

where the stimulated emission rate is

$$R_{\text{stim},\omega}(\mathbf{r}) = \frac{1}{[\varepsilon_g(\mathbf{r}) + \varepsilon_m(\mathbf{r})]} \frac{1}{3\pi\omega\varepsilon_0} \frac{e^2 P^2}{m_0^2} \left(\frac{2\mu_r}{\hbar^2}\right)^{3/2} (\hbar\omega - E_g)^{1/2} (f_c - f_v). \quad (31)$$

One can get a good order-of-magnitude estimate of the stimulated emission rate by using relation between the electron effective mass and momentum matrix element $m_0^2 / m_c = 2P^2 / \hbar\omega$,

$$R_{\text{stim},\omega}(\mathbf{r}) \approx \frac{4\alpha_0 c}{3[\varepsilon_g(\mathbf{r}) + \varepsilon_m(\mathbf{r})]} \sqrt{\frac{2\mu_r}{\hbar^2} (\hbar\omega - E_g) (f_c - f_v)}, \quad (32)$$

where $\alpha_0 = 1/137$ is a fine structure constant. Interestingly, the dependence on the material itself is rather weak. For the degenerately pumped semiconductors the square root is roughly equal to the Fermi wave vector, $k_F = (3\pi^2 n)^{1/3}$ and we obtain

$$R_{\text{stim},\omega}(\mathbf{r}) \approx \frac{4\alpha_0 (3\pi^2)^{1/3} c [\tilde{n}(\mathbf{r})]^{1/3}}{3[\varepsilon_g(\mathbf{r}) + \varepsilon_m(\mathbf{r})]} (f_c - f_v) \approx 1 \times 10^{14} \text{ s}^{-1} [\tilde{n}(\mathbf{r}) / 10^{19}]^{1/3} (f_c - f_v). \quad (33)$$

Note that if one can define the propagating mode inside the cavity with a group velocity v_g and introduce a power flow density $S_{\omega} \equiv v_g U_{\omega}$, one could find the material gain per unit length

$$g_{\omega}(\mathbf{r}) = v_g^{-1} R_{\text{stim}}(\mathbf{r}) \sim 1 \times 10^4 \text{ cm}^{-1} (\tilde{n} / 10^{19})^{1/3} (f_c - f_v). \quad (34)$$

However, for the mode in the sub-wavelength cavity one **cannot define the group (or, for that matter, phase) velocity, hence gain per unit length is essentially meaningless**, except that it can be used as a measure of population inversion for comparison with conventional semiconductor lasers.

It makes sense then to express the stimulated emission rate as an explicit function of carrier density by introducing the differential gain

$$G_{\text{stim},\omega} = \frac{R_{\text{stim},\omega}(\mathbf{r})}{\tilde{n}(\mathbf{r})} \quad (35)$$

which, according to (33), can be shown to remain relatively unchanged and on the scale of 10^{-5} cm³/s over the wide range of carrier densities, especially when these carrier densities are large, hence we can linearize the rate equations in the main text when we estimate frequency response of the spaser.

We now turn our attention to the rate of spontaneous emission which can be found handily from (30) assuming that one has just one polariton in the mode and noticing that spontaneous emission does not require population inversion and is proportional to $f_c(1 - f_v)$

$$-\frac{d\tilde{n}(\mathbf{r})_{\text{spont},\omega}}{dt} = \frac{\gamma R_{\text{stim},\omega}(\mathbf{r})u_\omega(\mathbf{r})}{V_{\text{eff}}} \quad (36)$$

where $\gamma \sim f_c(1 - f_v)/(f_c - f_v) > 1$ is the excess noise factor. We can now compare this spontaneous rate to that in the unconstrained dielectric in order to **define Purcell factor, even though, as we shall see, this factor is largely irrelevant when it comes to threshold.** In order to find out the total spontaneous rate in the absence of cavity, in place of $U_\omega(\mathbf{r})$ one should use the spectral density of vacuum fluctuations in the unconstrained semiconductor,

$$U'(\omega) = \hbar\omega \frac{\epsilon_r^3 \omega^2}{\pi^2 c^3} \quad (37)$$

and then perform integration over the frequencies, which leads to

$$-\frac{d\tilde{n}_{\text{spont},0}}{dt} = \frac{\epsilon_r^3}{\pi^2 c^3} \int_{E_g/\hbar}^{\infty} R_{\text{stim},\omega}(\mathbf{r})\gamma(\omega)\omega^2 d\omega \approx \frac{8\pi\gamma}{\lambda^3} \frac{\Delta\omega_{\text{sp}}}{\omega} R_{\text{stim},\omega}(\mathbf{r}) \quad (38)$$

where λ is the wavelength inside the semiconductor, and $\Delta\omega_{\text{sp}}$ is the linewidth of the spontaneous emission, defined as

$$\Delta\omega = \int f_c(\omega)(1 - f_v(\omega))d\omega / f_c(\omega_{\text{max}})(1 - f_v(\omega_{\text{max}})) \quad (39)$$

where ω_{max} is the angular frequency of the peak. Comparing (39) with (36) leads to the value of position-dependent Purcell factor

$$F_\omega(\mathbf{r}) = \frac{\lambda^3}{8\pi V_{\text{eff}}} \frac{\omega}{\Delta\omega_{\text{sp}}} u_\omega(\mathbf{r}). \quad (40)$$

Note that for a broad gain semiconductor medium, Purcell factor does not depend on the quality factor of the cold cavity itself (i.e. on the metal loss), but only on the spectral width of luminescence. The maximum value of Purcell factor obviously occurs near the surface of nanoparticle where $u_\omega(\mathbf{r}) = 1$. For most semiconductors that are degenerately pumped to about 10^{19} cm⁻³, the $\omega/\Delta\omega_{\text{sp}}$ factor is on the order of 10, hence for the characteristic mode size of $\lambda/10$ (40 nm mode diameter for telecommunication wavelength of 1320nm in InGaAs) the Purcell factor can be as high as 500, indicating spontaneous emission rate on the picosecond scale. **This number though is entirely irrelevant to all the results obtained in the main text except for the last part of this work where the device speed is considered.**

What remains to be done is to add stimulated (30) and spontaneous rates (36) and perform integration over the volume of the gain medium, yielding the total rate of emission into the mode

$$\frac{dn_p}{dt} = \int_{\text{gain}} \left[\frac{d\tilde{n}(\mathbf{r})_{stim,\omega}}{dt} + \frac{d\tilde{n}(\mathbf{r})_{spon,\omega}}{dt} \right] dV = V_{eff}^{-1} (n_p + \gamma) \int_{\text{gain}} R_{stim,\omega}(\mathbf{r}) u_\omega(\mathbf{r}) dV. \quad (41)$$

As one can see that the Purcell factor indeed is not explicitly evident in (41). In fact, assuming that gain volume occupies a fixed fraction, say half of the mode volume, the confinement factor and thus the whole (41) are volume independent. When effective volume gets smaller both spontaneous and stimulated recombination rates per electron/hole pair increase as V_{eff}^{-1} , but then multiplying by the gain medium volume cancels that dependence. We invoke (35) to obtain

$$\frac{dn_p}{dt} = \frac{\Gamma_\omega G_{stim,\omega} n_c (n_p + \gamma)}{V_{eff}} \quad (42)$$

where

$$\Gamma_\omega = \int_{\text{gain}} \tilde{n}_c(\mathbf{r}) u_\omega(\mathbf{r}) dV / \int_{\text{gain}} \tilde{n}_c(\mathbf{r}) dV \quad (43)$$

is the overlap factor between the carriers and polaritons which is close to unity and is different from the mode confinement factor as it is usually defined. All that is left is to introduce the effective rate of spontaneous emission into the lasing (spasing) mode at frequency ω as

$$\frac{1}{\tau_s} = \frac{R_{stim,\omega}}{\tilde{n}_c} \frac{\mathcal{A}\Gamma_\omega}{V_{eff}} \approx G_{stim,\omega} \frac{\mathcal{A}\Gamma_\omega}{V_{eff}} \quad (44)$$

and then include the rate of decay of the polaritons in the cavity τ_c^{-1} to obtain

$$\frac{dn_p}{dt} = \frac{1}{\tau_s} (\gamma^{-1} n_p + 1) n_c - \frac{n_p}{\tau_c} \quad (45)$$

which is, of course, the second of equations (4) in the main text.

To obtain the first equation in (4) one must take into account the fact that emission can also take place into the additional modes. For the semiconductor laser that is larger than wavelength these modes are additional cavity modes and also free space modes. For sub-wavelength spaser one can think about two additional dipole modes with different orientation (polarization) and also higher order modes. We then can write

$$\frac{dn_c}{dt} = \frac{I}{e} - \frac{1}{\tau_s} (\gamma^{-1} n_p + 1) n_c - \sum_m \frac{n_c}{\tau_{s,m}} \quad (46)$$

where the summation is over all the modes other than the spasing one. Then one can introduce

$$\beta = \frac{\tau_s^{-1}}{\tau_s^{-1} + \sum_m \tau_{s,m}^{-1}} \quad (47)$$

to obtain finally

$$\frac{dn_c}{dt} = \frac{I}{e} - \frac{1}{\tau_s} (\gamma^{-1} n_p + \beta^{-1}) n_c. \quad (48)$$

For the properly designed spaser one can minimize the rates of emission into the additional modes by using elliptical nano-particles so that the frequencies of those modes are blue shifted beyond the gain spectrum, and by employing spacers to avoid spontaneous emission into the higher order modes (quenching). Then, since the spontaneous emission into the lasing mode is Purcell enhanced β will be close to unity.

If β is less than unity and the volume is still sub-wavelength, then the threshold will be only more difficult to reach.

Note that the spontaneous emission is inversely proportional to the effective mode volume, and thus enhanced by the Purcell factor. But the total carrier density n_c is proportional to the volume of the cavity, hence these two dependencies cancel each other. Therefore, the only volume-dependent parameter in (45) is the decay rate of polaritons, which for substantially sub-wavelength volumes becomes volume-independent. In fact we can re-write (45) as

$$\frac{dn_p}{dt} = R_{stim,\omega} (n_p + \gamma) - \frac{n_p}{\tau_c} \quad (49)$$

to see that, the decay rate of polaritons is essentially the only property that determines the spaser characteristics.

As shown in the main paper the threshold of the spaser occurs when one gets $n_{p,th} = \gamma$ photons per mode, which quickly brings us to **the threshold value of stimulated emission rate of**

$$R_{stim,th} = \frac{1}{2\Gamma_c \tau_c} \approx 2 \times 10^{14} s^{-1} \quad (50)$$

where we assumed $\tau_c = 10 fs$, and the confinement factor

$$\Gamma_c = \frac{\int \tilde{n}(\mathbf{r}) u(\mathbf{r}) dV}{\tilde{n}_{max} V_{eff}} \quad (51)$$

was assumed to be on the order of 1/4 as confirmed by computer simulations.

Then, according to (33) the threshold value of carrier population is about on the order of $10^{19} cm^{-3}$ essentially the same as calculated in our previous works using exact numerical analysis [Refs 5,16]. The **gain per unit length** can be found from (34) as

$$g_{\omega,th} \sim v_g^{-1} R_{stim} \sim 0.8 \times 10^4 cm^{-1}. \quad (52)$$

By themselves, neither threshold gain nor threshold carrier densities are outrageously high, but the current required to maintain their values is huge for such small volumes. Finally, according to (44) and (50) the spontaneous lifetime at threshold can be found as

$$\frac{1}{\tau_s} = \frac{R_{stim,th}}{\tilde{n}_{c,th}} \frac{\gamma \Gamma_\omega}{V_{eff}} \approx \frac{\gamma}{2n_c \tau_c}. \quad (53)$$

As one can see the threshold carrier lifetime is inversely proportional to the total number of carriers in spasers – the smaller is the volume the less is the number of carriers and hence the spontaneous lifetime decreases. Of course, the physical reason for this shortening of the lifetime is the Purcell factor, but one can estimate this lifetime to be on the order of

$$\tau_s = 2\tau_c V_{gain} \tilde{n}_{th} / \gamma \approx 10^{-16} s \cdot V_{gain} \quad (54)$$

(where volume is in nm³) without actually calculating the Purcell factor or gain. Clearly, there is no wavelength dependence in (54) for as long as the metal loss stays wavelength-independent. For 30nm radius mode with a half of it occupied by gain medium we obtain $\tau_s \sim 5ps$.

One can summarize the role of Purcell factor in the following statement. In a single-mode spaser the value of cavity loss must be exactly compensated by the rate of stimulated emission, which at threshold, is equal to the rate of spontaneous emission. This fact alone determines the spontaneous emission time. The mechanism for the enhancement of the spontaneous decay rate is Purcell effect, but one need not to know the Purcell factor in order to find threshold current.