Supplemental Information for

Nonlinear Abbe Theory

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This Supplemental Information discusses aperture effects, signal properties as a function of nonlinear strength and type, and the extension to non-paraxial behavior. It is divided into three parts. Part 1 demonstrates the effects, and advantages, of using an apodized aperture in spatially band-limited systems. Part 2 compares Kerr and saturable nonlinearities and highlights the issue of strong wave mixing in single-shot reconstruction. Finally, Part 3 discusses the nonlinear coupling of evanescent and propagating waves. Both analytic and numerical results are given.

1. Aperture effects

In the experiments, a hard aperture in the image plane was used. In the frequency domain, a hard edge corresponds to a convolution of the signal spectrum with a sinc function (or with a jinc function in a 2D circular geometry). Numerical deconvolution with this function results in low-amplitude oscillations in the final reconstructed images (seen in both the nonlinear and linear cases). It is possible to correct for this in post-processing, similar to other deconvolution techniques, though the zeros in the transfer function often introduce significant noise. Alternatively, the issue can be solved physically by using an apodized aperture. In Fig. S1, we demonstrate the differences in reconstructions between using a rectangular window and a super-gaussian aperture. In the latter case, oscillations in the reconstructed image are virtually eliminated. Note that the reconstruction is significantly improved even though the Fourier transform of the apodized aperture does not differ greatly from the original one (Fig. S1e).

![Figure S1: Nonlinear output and window function for a) hard and b) apodized apertures. c) d) Reconstruction results (green) compared to object input (blue dashed). e) Spatial spectra of (red) hard and (black) apodized apertures.](image)

2. Effects of nonlinearity strength and form

From a systems perspective, nonlinear propagation may be considered as an intensity-dependent transfer function. Object modes evolve, and new spatial frequencies are created, in response to the nonlinear medium. The choice of an appropriate nonlinear form and strength becomes a new design challenge in imaging optics, analogous to similar choices in the linear case, e.g. point-spread function engineering for super-resolution [S1], phase masks for uniform focus [S2], or coded apertures for depth estimation [S3]. Because the transfer function is now dynamical, rather than fixed, the optimum result occurs when the nonlinear response is matched with the system response. This resonance-like behavior is separate from the usual imaging trade-offs from linear theory, e.g., a larger aperture increasing resolution but reducing depth of field, so that new optima are possible. For example, objects of interest often belong to a known class of signals; this a priori information can be used to find the best nonlinear material (in much the same way as such priors determine which basis to use in compressed sensing approaches [S4]).
2A. Nonlinear strength

A numerical demonstration of the imaging resonance is given in Fig. S2, which shows a set of simulated nonlinear reconstructions for a three-bar input profile. As the nonlinearity is increased, with all other parameters fixed, energy collects in the side bars. New spatial frequencies are generated inside the spectral transmission window (Fig. S3), resulting in a more accurate reconstruction. However, further increases in nonlinearity result in too much modal coupling and spectral beating, leading to nonlinear modes which overpower the aperture (system) transfer function. Simple back-propagation of the output leads to distortion effects (here, a reconstructed width that is larger than the input widths). Deconvolution with the aperture function helps, but errors accumulate as the nonlinear modes cascade. The solution, as in structured illumination, is to increase the spectral range sequentially, using previous reconstructions to remove any ambiguity.

**Figure S2:** Nonlinear output and reconstruction of three-bar input for increasing nonlinear strength. The inset number in each panel is the dimensionless ratio of linear to nonlinear length scales, \( k_2^2 / (\gamma L_0) = L_D / L_{NL} \). Scale bar: 80 µm.

**Fig. S3:** Associated spatial spectra of the nonlinear outputs in Figure S2. The inset number in each panel is the dimensionless ratio of linear to nonlinear length scales, \( k_2^2 / (\gamma L_0) = L_D / L_{NL} \). Scale bar: 0.1 µm⁻¹.
2A. Nonlinear form

To illustrate the material dependence, we show in Figs. S4 and S5 reconstructions for both Kerr and saturable nonlinearities, \( \Delta n = \gamma I/(1+I/I_{\text{sat}}) \), for both a three-bar resolution target and a quasi-random input signal. For the three-bar input, saturation yields two salient features. First, it tends to weaken the distortion (at the expense of the reconstructed signal strength). Second, less energy leaves the central peak. For the quasi-random input, saturation tends to change the feature sizes that can be reconstructed. The optimal material response, both in general and for particular objects, remains an open question.

![Figure S4](image1.png)

**Figure S4:** Results for Kerr (a) and saturable (b-d) nonlinearities, \( I_{\text{sat}} = 4I_{\text{max}}, 2I_{\text{max}}, \) and \( I_{\text{max}} \). For each panel, the top row represents the input (dotted) and reconstructed input, and the bottom row corresponds to the nonlinear output. Red bars indicate aperture width.

![Figure S5](image2.png)

**Figure S5:** Same as in Fig. S4 for a quasi-random input.
3. Paraxial to nonparaxial coupling

Nonlinear wave mixing is valid at any spatial scale, including regimes that are well beyond the paraxial limit. Here, we are interested in imaging in the far field, when propagating modes are observed in the linear case but evanescent waves (sub-wavelength features) are not. To illustrate the problem in its most basic form, we consider the coupling between a single evanescent wave \( E_1 \) and a propagating wave \( E_2 \). For simplicity, we ignore polarization and use only the scalar nonlinear wave equation:

\[
\left( \nabla^2 + k_0^2 \right) E + 2k_0^2 \left( \frac{n_2}{n_0} \right) |E|^2 E = 0. \tag{S1}
\]

where \( k_0 = \frac{2\pi}{\lambda} \) is the material wavenumber, \( n_0 \) is the base index of refraction of the medium, and \( n_2 \) is the Kerr coefficient measuring the strength of the cubic nonlinearity. For the self-defocusing case considered here, \( n_2 < 0 \).

3A. Analytic results

Writing

\[
E = \sum_{j=1}^{2} A_j(z) e^{ik_j r} \tag{S2}
\]

yields

\[
A_j'' + 2ik_j A_j' + 2k_0^2 \left( \frac{n_2}{n_0} \right) \left( I_j + 2I_1 \right) A_j = 0, \tag{S3}
\]

where \( \{j,l\} = \{1,2\}, j \neq l \). Here, the primes refer to derivatives with respect to the propagation distance \( z \). To simplify the problem further, we let \( A_2 \) be a paraxial propagating wave but keep \( A_1 \) general. This allows us to make the two approximations \( k_{j,l} \sim 0 \) and \( d^2A_2/dA_2^2 \sim 0 \). The system then reduces to the coupled wave equations

\[
A_1'' + 2k_0^2 \left( \frac{n_2}{n_0} \right) \left( I_1 + 2I_2 \right) A_1 = 0 \tag{S4a}
\]

\[
2ik_{2z} A_2' + 2k_0^2 \left( \frac{n_2}{n_0} \right) \left( I_2 + 2I_1 \right) A_2 = 0 \tag{S4b}
\]

Equation (S4b) assumes that the variation of \( A_2 \) varies on a length scale that is long compared with a wavelength, \textit{i.e.} it is effectively an undepleted pump beam for two-wave mixing. To see this explicitly, we use the polar transformation \( A_2 = a_2 e^{i\varphi_2} \) to separate Eq. (S4b) into real and imaginary components:

\[
2k_{2z} a_2' = 0 \tag{S5a}
\]

\[
-2k_{2z} q_2' + 2k_0^2 \left( \frac{n_2}{n_0} \right) \left( I_2 + 2I_1 \right) = 0 \tag{S5b}
\]

Equation (S5a) reveals that \( a_2 \) is a constant, and the phase change \( \varphi_2 \) depends on only \( a_1 \):

\[
q_2' = \frac{k_{2z}^2}{k_0^2} \left( \frac{n_2}{n_0} \right) \left( I_2 + 2I_1 \right) \tag{S6}
\]

where \( I_2 = |a_2|^2 \). The equation for \( A_1 \) follows directly from Eqn. (S4a) and a similar polar substitution:
\[2ia'_i q'_i + ia q'_i^* = 0\]  
(S7a)

\[a'_i - a_i \left( \frac{C_i}{a_i^2} \right)^2 + 2k_0^2 \left( \frac{n}{n_0} \right) a_i^2 + 2a_i^2 \lambda_i = 0\]  
(S7b)

The first equation can be integrated directly to give

\[a_i^2 q'_i = C_i\]  
(S8)

where \(C_i\) is the constant of integration substituted into Eqn. (S7b). With integration by quadrature, we arrive at a nonlinear ordinary differential equation for \(a_i\):

\[(a'_i)^2 + C_i^2 a_i^{-2} + 2k_0^2 \left( \frac{n}{n_0} \right) \left( \frac{1}{2} a_i^4 + 2a_i^2 a_i^2 \right) = C_2\]  
(S9)

where \(C_2\) is another constant of integration. Noting that \(a_i^2 = I_1\), Eqn. (S9) can be rewritten as

\[\frac{(I'_1)^2}{4I_1} + C_i^2 I_1^{-1} + 2k_0^2 \left( \frac{n}{n_0} \right) \left( \frac{1}{2} I_1^2 + 2I_1 I'_1 \right) = C_2\]  
(S10)

We now scale to dimensionless variables by letting \(\xi \equiv I_1/I_2\) let \(\zeta \equiv k_0\zeta\). The constants of integration become

\[\frac{C_1}{k_0 I_2} = \xi \varphi'_{1}\zeta = \sqrt{K_1}\]  
(S11a)

\[\frac{C_2}{I_2 k_0^2} = \frac{(\xi \zeta)^2}{4\xi} + \xi \varphi'_{1\zeta} + \left( \frac{n f_{1z}}{n_0} \right) (\xi^2 + 4\xi) = K_2\]  
(S11b)

so that the central equation becomes:

\[\pm 2d\zeta = \frac{d\xi}{\sqrt{-K_1 + K_2 \zeta - 4 \left( \frac{n f_{1z}}{n_0} \right) \zeta^2 - \left( \frac{n f_{1z}}{n_0} \right) \zeta^3}}\]  
(S12)

Solutions are elliptic functions, which should be expected given the oscillatory nature of spectral beating.

As a convenient limit for evaluating Eqn. (S12), we assume weak nonlinearity, i.e. \(\delta = -n_2 I_2/n_0 \ll 1\) (with \(n_2\) negative for defocusing), so that the cubic term in the radicand can be ignored. The result is

\[\zeta = \frac{K_2}{8\delta} \left( -1 + \sqrt{\frac{16K_1}{K_2} + \frac{K_2^2}{K_2^2}} \cosh \left( 4\sqrt{\delta} k_0 z \right) \right)\]  
(S13)

Note that in this approximation, \(\zeta \equiv I_1/I_2\) grows exponentially with distance.
To calculate the phase change for \( \phi_2 \), using Eq. (S13) in Eq. (S6), we approximate \( K_1 \) and \( K_2 \) by noting that the phase change for mode \( A_1 \) is rapid, so that its derivative over a the distance of a wavelength is \( \sim 2\pi \). For \( \xi_0, \delta \ll 1 \),

\[
K_1 = (2\pi \xi_0)^2 \\
K_2 = (2\pi)^2 \xi_0
\]  

(S14a)

(S14b)

We can now solve Eq. (S6) for \( \phi_2 \):

\[
\frac{\phi_2(z) - \phi_2(0)}{\Delta \phi_2(z)} = -\frac{k_0^2}{k_{2z}^2} \delta \left\{ z + \xi_0 \frac{\pi^2}{\delta} \left\{ -z + \sqrt{1 + \frac{4\delta}{\pi^2} \sinh \left( \frac{4\sqrt{\delta} k_0 z}{\pi} \right)} \right\} \right\} 
\]

(S15)

Letting \( k_{2z} \sim k_0 \) and \( z = \lambda \) gives

\[
\Delta \phi_2(\lambda) = -2\pi \delta \left\{ 1 + \xi_0 \frac{\pi^2}{\delta} \left\{ -1 + \sqrt{1 + \frac{4\delta}{\pi^2} \sinh \left( \frac{8\pi \sqrt{\delta}}{8\pi \sqrt{\delta}} \right)} \right\} \right\}
\]

(S16)

which can be expanded to yield

\[
\Delta \phi_2(\lambda) = -2\pi \delta \left( 1 + 2\xi_0 \right)
\]

(S17)

For nonlinear materials such as organic polymers, \( \delta \sim 0.01 \); assuming \( \xi(0) = I_0(0)/I_1 \sim 0.1 \), the phase change is \( -2\pi \times 0.01 \times (1 + 1.68) = -0.02\pi \sim -0.033\pi \). The second term is the phase change corresponding to the presence of the nonparaxial wave and, in this case, represents a 150% increase in phase accumulation. This term is roughly \( \lambda/60 \), a difference which can be detected, e.g. through dual-wave holo- 
graphy [S5]. Therefore, for suitably chosen materials, one can measure information about highly nonparaxial modes through phase measurements.

### 3B. FDTD simulation

Figure S6 shows finite-domain time-difference simulation (MIT MEEPS) of the two-wave mixing problem described in the previous section. A propagating wave is coupled with an evanescently decaying wave, with initial field distribution

\[
A(x,t) = A_0 \left[ I + \sin(2k_0 x) \right] e^{-|x|/\lambda_0}
\]

where \( k_0 = 2\pi/\lambda \). Consistent with the symmetry of the source and material, we assume that both the pattern and the optical response satisfy periodic boundary conditions along the x-axis. The left and right ends of the z-axis have perfectly matched layers to absorb signals. To assure a steady-state regime, data is time-averaged over several oscillations of the field.

Shown in Fig. S6 is the time-averaged power spectrum at a propagation distance of \( z = 9\lambda \) from the source. At this distance, the original evanescent mode has decayed to \( 2^{-9} \sim 10^{-4} \) of its original intensity. With nonlinearity, this mode can be orders of magnitude stronger. Further, new modes appear at the spatial harmonic \( 2\pi/\lambda/4 \).
**Figure S6.** FDTD simulation of power spectrum at z = 9λ for nonlinear coupling of propagating [k_x = 0] and evanescent [k_x = 2π/(λ/2)] waves.

### Supplemental References


