Biological measurement beyond the quantum limit: 
Supplementary Information

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1 The quantum noise limit to position sensing

1.1 Derivation of the limit to position sensing

Here, the quantum noise limit for optical position sensing is derived. The optical electric field amplitude $\hat{E}$ at position $X$ on the detector can be decomposed into a collection of modes,

$$\hat{E}(X) = \sum_{n=0}^{\infty} \hat{E}_n(X) = i \sqrt{\frac{\hbar \omega}{2 \varepsilon c}} \sum_{n=0}^{\infty} \hat{a}_n \psi_n(X) z_n,$$

where the $n$th mode has a normalized mode shape of $\psi_n$, polarization $z_n$ and the annihilation operator $\hat{a}_n$. This operator can be expressed as a mean value $\alpha_n$ and a fluctuating operator $\hat{\delta a}_n$,

$$\hat{a}_n = \alpha_n + \hat{\delta a}_n.$$

The photon number $n$ of the mode is given by $n_n = |\alpha_n|^2$. We also define quadratures in the usual way as

$$\hat{X}_n^+ = (\hat{\delta a}_n + \hat{\delta a}_n^\dagger),$$

$$\hat{X}_n^- = -i(\hat{\delta a}_n - \hat{\delta a}_n^\dagger),$$

$$\hat{X}_n^\theta = e^{i\theta} \hat{\delta a}_n + e^{-i\theta} \hat{\delta a}_n^\dagger = \cos \theta \hat{X}_n^+ - \sin \theta \hat{X}_n^-.$$

For any of these quadratures, the fluctuation statistics for coherent, quantum noise limited modes and unoccupied vacuum modes are given by

$$\langle \delta \hat{X}_n(\omega) \rangle = 0$$

$$\langle (\delta \hat{X}_n(\omega))^2 \rangle = 1$$

Squeezed states violate Eq. (7), and instead have lower variance on one quadrature at the expense of the other. The variances of a phase or amplitude squeezed state can be represented as

$$\langle (\delta \hat{X}_n^\pm(\omega))^2 \rangle = V^\pm,$$

where $V^\pm$ is the variance of the quadrature. These can individually be less than 1, provided the Heisenberg condition is met:

$$V^+ V^- \geq 1.$$
When fields are detected with an efficiency of $\eta$, this is equivalent to placing a beamsplitter in the path with transmission $\eta$. This couples out some of the occupation of the mode, while introducing some vacuum fluctuations. The measured field is thus given by

$$\alpha_n^{\text{det}} + \delta a_n^{\text{det}} = \eta^{1/2} \alpha_n + \eta^{1/2} \delta a_n + (1 - \eta)^{1/2} \delta a_n^0$$  \hspace{1cm} (10)

where $\delta a_n^0$ is the vacuum fluctuation. If a detector with spatial structure is used, such as a quadrant photodiode, then this influences the measured signal. The spatial variation in the gain of the detector is described by $U(X)$; for a bulk detector, this is always 1, while for split detection, this is -1 on the left side and 1 on the right side.

The occupied optical modes at the detector are from the local oscillator $E_{\text{LO}}$, scattered $E_{\text{scat}}$, and some residual trapping field $E_T$. The vacuum fluctuations present in all other modes are orthogonal to both of these ($\langle \delta a \delta a^\dagger \rangle = 0$). To minimize the effect of the trap field on detection, it was made orthogonal in both frequency and polarization to the local oscillator and scatter. Including detector inefficiency, the detected signal photocurrent $I$ of this measurement is given by

$$I = \frac{2\varepsilon}{\hbar \omega} \int_{-\infty}^{\infty} U(X) \sum_{n=0}^{\infty} \tilde{E}_{n,\text{det}}^{\text{det}} |^2 dX$$

$$= \frac{2\varepsilon}{\hbar \omega} \int_{-\infty}^{\infty} U(X) |\tilde{E}_{\text{LO}}^{\text{det}} + \tilde{E}_{\text{scat}}^{\text{det}} + \tilde{E}_T^{\text{det}}|^2 dX$$

$$= \frac{2\varepsilon}{\hbar \omega} \int_{-\infty}^{\infty} U(X) |\tilde{E}_{\text{LO}}^{\text{det}} + \tilde{E}_{\text{scat}}^{\text{det}}|^2 + U(X) |\tilde{E}_T^{\text{det}}|^2 dX.$$  \hspace{1cm} (11)

The separation of the trap from other fields, and the dropping of the polarization vector are because the local oscillator and scattered field share polarization, while the trap is orthogonal to both of these ($\langle \tilde{E}_{\text{LO}} \cdot \tilde{E}_T \rangle = 0$).

It has also been assumed here that the local oscillator mode $\tilde{E}_{\text{LO}}^{\text{det}}$ is not affected by the particle. Without loss of generality, the local oscillator amplitude $\alpha_{\text{LO}}$ is thus set to be real. The particle position is measured via the scattered field $E_{\text{scat}}$. This field is produced as light scatters out of an incident “probe” field. The number of scattered photons may be expressed in terms of the particles scattering cross-section $\sigma_{\text{scat}}$, the incident photon flux $n_{\text{incident}}$, and the incident beam width $w$ as

$$n_{\text{scat}} = \sigma_{\text{scat}} n_{\text{incident}} / 4\pi w^2$$  \hspace{1cm} (12)

Conventionally, it is scattering from the local oscillator which is measured. In that case, $n_{\text{incident}} = n_{\text{LO}}$. For our setup, we used a separate probe which had been amplitude modulated, such that the incident photons were in side-bands of the laser optical frequency. This resulted in scattered photons which were likewise in these side-bands, although this is not explicitly shown in the following equations. To find the dependence of the scattered field on a small particle displacement $x$, it can be expanded to

$$E_{\text{scat}} = \tilde{E}_{\text{scat}} |_{x=0} + x \frac{d\tilde{E}_{\text{scat}}}{dx} |_{x=0}.$$  \hspace{1cm} (13)

The component $\tilde{E}_{\text{scat}} |_{x=0}$ is symmetric so has no overlap with the antisymmetric detection mode, and in the limit $\alpha_{\text{scat}} \ll \alpha_{\text{LO}}$ is not detected; $\int U(X) |\tilde{E}_{\text{LO}} \tilde{E}_{\text{scat}} |_{x=0} | dX = 0$. All particle position information is within the component $x \frac{d\tilde{E}_{\text{scat}}}{dx} |_{x=0} = \alpha_{\text{scat}} \psi_{\text{scat}} ^\prime x$, where $\psi_{\text{scat}} ^\prime = \frac{d\psi_{\text{scat}}}{dx} |_{x=0}$, so this defines the ideal measurement mode shape. This mode has been calculated for Rayleigh scattering particles\cite{1,2}, and is approximately given by the TEM01 mode. The actual detection mode, however, is given by $\psi_{\text{det}} = U(X) \psi_{\text{LO}}$. For our case, we use a bulk detector for which $U(X) = 1$ and thus $\psi_{\text{det}} = \psi_{\text{LO}}$ (note that this is not the case for split detection). Using this in
In the limit that the information is that a split detector has a detection mode \( \psi_{\text{det}} \) with a phase plate into the Fourier transform of a flipped Gaussian. In our setup, the mode is shaped with a phase plate into the Fourier transform of a flipped Gaussian. In the limit that the information mode is a TEM01 mode, these two schemes have equivalent sensitivity, although our setup allows optimal

\[
I = |\eta^{1/2} a_{LO} + \eta^{1/2} \delta a_{LO} + (1 - \eta)^{1/2} \delta a_{LO}^0|^2 \\
+ |\eta^{1/2} a_T + \eta^{1/2} \delta a_T + (1 - \eta)^{1/2} \delta a_T^0|^2 \\
+ 2\eta a_{LO} a_{\text{scat}} \text{Re}\{\psi'_{\text{scat}}|\psi_{\text{det}}\}
\]  

(16)

Here \( \langle \psi'_{\text{scat}}|\psi_{\text{det}}\rangle = \int_{-\infty}^{\infty} \psi_{\text{det}}^* \psi'_{\text{scat}} dX \). The expectation value for the photocurrent is then found to be

\[
\langle I \rangle = \eta m_{LO} + \eta m_T + 2\eta a_{LO} a_{\text{scat}} \langle x \rangle \text{Re}\{\psi'_{\text{scat}}|\psi_{\text{det}}\}
\]

(17)

which uses \( \langle x \rangle = 0 \). Taking the Fourier transform of Eq. 17, and calculating the variance, we find

\[
\langle I^2(\omega) \rangle = \eta^2 n_{LO}^2 \delta(\omega) + \eta m_{LO} \left( \eta \langle \delta \hat{X}_{LO}^+ (\omega) \rangle^2 + (1 - \eta) \langle \delta \hat{X}_{LO}^0 (\omega) \rangle^2 \right) \\
+ \eta^2 n_T^2 \delta(\omega) + \eta m_T \left( \eta \langle \delta \hat{X}_T^\theta (\omega) \rangle^2 + (1 - \eta) \langle \delta \hat{X}_T^\theta (\omega) \rangle^2 \right) \\
+ 4\eta^2 n_{LO} n_{\text{scat}} \langle x^2(\omega) \rangle \text{Re}\{\psi'_{\text{scat}}|\psi_{\text{det}}\}^2
\]

(19)

where the small cross terms such as \( \langle \delta \hat{X}_{LO}^+ \delta \hat{X}_T^\theta \rangle \) have been neglected. The terms at zero frequency \( \eta^2 n_{LO}^2 \delta(\omega) \) and \( \eta^2 n_T^2 \delta(\omega) \) can be neglected as they have no significance to the measurement. Further, since the trap beam is in a coherent state, \( \langle \delta \hat{X}_T^\theta \rangle^2 = 1 \). We define the probe beam variance as \( \langle \delta \hat{X}_{LO}^+ \rangle^2 \) = \( V \), where \( V = 1 \) for classical measurements, and \( V < 1 \) when the probe is amplitude squeezed. Using these substitutions, we find

\[
\langle I^2(\omega) \rangle = \eta m_{LO} (\eta V + (1 - \eta)) + \eta m_T + 4\eta^2 n_{LO} n_{\text{scat}} \langle x^2(\omega) \rangle \text{Re}\{\psi'_{\text{scat}}|\psi_{\text{det}}\}^2
\]

(21)

The quantum noise limit is reached when \( V = 1 \), with a photocurrent variance given by

\[
\langle I^2(\omega) \rangle_{\text{QNL}} = \eta m_{LO} + 4\eta^2 n_{LO} n_{\text{scat}} \langle x^2(\omega) \rangle \text{Re}\{\psi'_{\text{scat}}|\psi_{\text{det}}\}^2
\]

(22)

where trap photons have been neglected. The displacement \( x \) is resolvable when the noise term (\( \eta m_{LO} \)) is equal to the signal (the last term in Eq. 22), giving the quantum noise limit

\[
\langle x^2(\omega) \rangle_{\text{QNL}} = \frac{1}{4\eta m_{\text{scat}} \text{Re}\{\psi'_{\text{scat}}|\psi_{\text{det}}\}^2}
\]

(23)

Including the trap photons, and allowing squeezing, the minimum resolvable displacement then equals

\[
\langle x^2(\omega) \rangle_{\text{min}} = \frac{n_{LO} (\eta V + (1 - \eta)) + n_T}{n_{LO}} \langle x^2(\omega) \rangle_{\text{QNL}}
\]

(24)

This equation is valid both for our detection scheme and the conventional split detection. The primary difference is that a split detector has a detection mode \( \psi_{\text{det}} \) usually given by a “flipped” Gaussian, while for our setup, this mode is shaped with a phase plate into the Fourier transform of a flipped Gaussian.
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Figure 1: This shows a simple setup in which the quantum noise limit derived here for particle tracking reduces to the usual quantum noise limit for phase estimation shaping of the local oscillator which will always outperform the split detection. Additionally, our method enables squeezing to be introduced far more easily to the setup than split detector allows, because here the detection mode and local oscillator are identical. This means that to detect squeezing, the mode shape of the squeezing must match the local oscillator mode. For split detection, the detection and local oscillator modes are spatially orthogonal, so introducing squeezing which matched the local oscillator mode would have no influence on the measured signal.

In addition to this, for our experiments the particle was illuminated with AM light, and consequently scattered AM light. The particle motion then adds a slow modulation to this fast AM. To measure this, the detected signal is demodulated at the frequency of the AM. Consequently, the electronic and optical noise which enters the measurement is that which was around the modulation frequency $\omega_m$, rather than the usual low-frequency noise.

$$\langle x^2(\omega) \rangle_{\text{min}} = n_{\text{LO}} \left( \eta V(\omega + \omega_m) + (1 - \eta) \right) + n_{\text{LO}} \langle x^2(\omega) \rangle_{\text{QNL}}$$

This allows all low-frequency noise to be evaded, which both improves the classical sensitivity, and allows squeezed light to be used without requiring the squeezing band to overlap with the frequency range of interest. While this technique is similar to the optical lock-in detection used in fluorescent microscopy, and has been proposed for sensing with squeezed light, there have been no previous experimental demonstrations of it either with squeezed light or optical tweezers.

1.2 Relation to the quantum noise limit for phase estimation

The expression in Eq. 23 generalizes the quantum noise limit imposed in phase estimation interferometry to situations in which information is included in spatial changes as well as phase shifts. For the case where no information is contained in the spatial profile, the limit in Eq. 23 reduces to the usual phase estimation quantum noise limit. Such a situation occurs if the motion $x$ is movement of an interferometer mirror, as shown in Fig. 1, with the “scattered” field given by a phase modulated input mode $\psi_{\text{in}}$.

$$\psi_{\text{scat}} = \psi_{\text{in}} e^{ikx}.$$  \hspace{1cm} (26)

With this definition of the scattered mode,

$$\psi_{\text{scat}}' = \frac{d\psi_{\text{scat}}}{dx} \bigg|_{x=0} = (ik)\psi_{\text{in}} e^{ikx} \approx (ik - k^2 x)\psi_{\text{in}}.$$  \hspace{1cm} (27)

Choosing the optimum phase for the detection mode, one has $\text{Re}\{\langle \psi_{\text{scat}}'|\psi_{\text{det}}\rangle\} = k\langle \psi_{\text{in}}|\psi_{\text{det}}\rangle = k\eta_{\text{homo}}^{1/2}$, where $\eta_{\text{homo}}$ is the overlap efficiency between the incident signal field and the local oscillator field. Substituting this into Eq. 23, we find the quantum noise limit given by

$$\langle x^2(\omega) \rangle_{\text{QNL}} = \frac{1}{4\eta \eta_{\text{homo}} n_{\text{scat}} k^2}. \hspace{1cm} (28)$$
As the phase being measured is given by $\phi = kx$, Eq. 28 can be rearranged to

$$\langle \phi(\omega)^2 \rangle_{\text{QNL}} = \frac{1}{4\eta_\text{homo} n_\text{scat}} = \frac{1}{4n_\text{det}},$$

(29)

where $n_\text{det}$ is the number of scattered photons arriving in the homodyne detector in the correct detection mode $\psi_\text{det}$. This matches the quantum noise limit for phase sensing. An alternative limit of $\langle \phi(\omega)^2 \rangle_{\text{QL}} = 1/n_\text{det}$ is used when the total photon number is constrained, rather than the signal photon number, with these photons divided equally between the signal and local oscillator fields. In our case, since the measurements are ultimately constrained by optical damage in the biological samples, a bright local oscillator may be used and Eq. 29 is the relevant limit.

### 1.3 Relationship to the standard quantum limit

In addition to the quantum noise limit derived here, the standard quantum limit characterizes the ultimate limit on sensitivity classically achievable given the number of photons. This limit would be reached with a perfect measurement ($\eta = \eta_\text{homo} = 1$) with coherent light and 100% efficient detection. From Eq. 29, this arrives as the well known usual expression for the standard quantum limit of a phase measurement $\langle \phi(\omega)^2 \rangle_{\text{SQL}} = (4n_\text{scat})^{-1}$. When this limit is overcome, it demonstrates that the experiment operates in a regime which is fundamentally inaccessible with classical resources, and this is typically used as a benchmark in quantum metrology experiments which involve discrete counting of small numbers of photons. By contrast, the quantum noise limit is used to characterize the improvement which quantum resources confer to a measurement with a given apparatus, and is relevant in determining the practical benefit of a quantum enhancement. This is generally used in experiments similar to ours, where bright optical fields are used for continuous measurements.

Due to the stringent requirements on measurement apparatus and detection efficiency it is challenging, in general, to surpass the standard quantum limit. This is particularly the case for microparticle tracking experiments, where the scattered field has complex spatial structure and is not well spatially confined. The detection apparatus must be able to near-optimally extract information from this complex spatial mode over $4\pi$ steradians. As far as the authors are aware, even the best classical microparticle tracking experiments remain more than a factor of 1000 away from the standard quantum limit.

### 2 Experimental details

#### 2.1 Squeezing setup

A bright amplitude squeezed laser beam is produced in an optical parametric amplifier (OPA) operating in a de-amplification regime. A dual wavelength laser at 1064 nm and 532 nm was used to drive the experiment. The OPA is of a bowtie cavity design with a periodically poled KTP crystal providing the nonlinear interaction. In order to maximise the cavity escape efficiency, faces of the nonlinear crystal were anti-reflection coated using the ion beam sputtering technique, to provide ultra-low loss for the intra-cavity circulating field. The cavity mirrors were also custom made with reflectivity exceeding 99.95%.

The squeezed field exits the cavity through an output-coupler with 10% transmission. The OPA is seeded with a bright laser beam such that bright amplitude squeezed light is produced. We used a single high quantum efficiency photodiode to measure squeezing, where the bright squeezed laser field acts also as a local oscillator in the detection process. The detector was calibrated against the shot noise using a QNL laser field. The squeezing level we achieved was of approximately -6 dB below the QNL at the sideband detection frequency 3.5 MHz. For performing tracking measurements of particles (either beads or yeast cells) where the information of the particle movement is contained in a well defined spatial mode, the Gaussian profile of the squeezed field was transformed into a flipped mode using a spatial phase plate which imparts $\pi$ phase shift onto one half of the beam.
2.2 Optical losses

Squeezing degrades sharply with added loss, as loss is an inherently random process. To avoid this, high efficiency optics were used for the optical tweezers, including low loss objectives (OFR-LMH-20X-YAG). As these have a relatively low numerical aperture of 0.4, they also impose minimal spatial distortion to the optical modes. The local oscillator encountered 19% loss in the optical setup, with 16% of this at the optical trap, and a further 3% at the phase plate. The objectives together contributed 7% loss, which is within the rated transmission of 96–98%. Most commercially available objectives have much more loss than this; for instance the Zeiss Objective A-Plan 20x/0.45 has around 85 % transmission at 1064 nm. The sample was suspended in a 120 µm thick water chamber between two glass slides. The loss in the chamber was minimized by anti-reflection coating both of the air-glass interfaces, such that the chamber itself added around 5% loss, with a further 4% lost from a trapped glass bead. Numerical calculations with Mie theory confirm that this loss results primarily from scattering of the field, rather than absorption. After the optical trap, the remaining 81 µW of local oscillator light was measured on a high efficiency, home-made bulk detector. This detector has efficiency > 95%, and at the optical power used, the electronic noise floor was neglected as it was determined to be 14 dB below the shot noise level. Hence, the local oscillator was detected with an efficiency of $\eta_{LO} = 0.76$ when tracking the silica beads. In the biological experiments, the yeast cells in the optical trap caused 9% more loss than the beads did, such that the local oscillator detection efficiency was $\eta_{LO} = 0.67$. This efficiency $\eta_{LO}$ characterizes the loss which the squeezed local oscillator experiences, and therefore determines the maximum quantum enhancement possible. However, the local oscillator photons carry no information about the particle position; this information is carried by scattered photons, and these photons are only sensitive to loss between the scattering particle and the detector. Thus, the scattered photons which leave the yeast cell or silica bead are measured with a higher efficiency of approximately of $\eta = 0.89$.

References