Direct determination of spin–orbit interaction coefficients and realization of the persistent spin helix symmetry

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Supplementary Notes

I. Theoretical approach using recursive Green’s function

In our numerical approach we use a finite difference scheme to represent the Hamiltonians $H_R$ and $H_D$ of the Rashba and the linear Dresselhaus SOI

\[ H_R = \frac{\alpha}{\hbar} \left( \sigma_x p_y - \sigma_y p_x \right), \quad (S1) \]

\[ H_D = \frac{\beta}{\hbar} \left[ \left( \sigma_x \cos 2\varphi - \sigma_y \sin 2\varphi \right) p_x - \left( \sigma_x \sin 2\varphi + \sigma_y \cos 2\varphi \right) p_y \right], \quad (S2) \]

on a quadratic lattice. Here $\sigma$ are the Pauli spin matrices and $p_i$ ($i = x, y$) is the in-plane momentum. The $x, y$ directions are chosen parallel and perpendicular with the narrow wire direction, respectively. $\varphi$ denotes the angle between $x$ direction and [100] direction of the zinc blende crystal.

The relation between the SOI parameters ($\alpha, \beta$) and the direction of effective magnetic field ($B_{\text{eff}}$) $\theta_{\text{eff}}$ can be expressed by following equation\textsuperscript{31}.

\[ -\frac{\alpha \cos \varphi + \beta \sin \varphi}{\alpha \sin \varphi + \beta \cos \varphi} = \tan \theta_{\text{eff}}. \quad (S3) \]

The angle $\theta_{\text{eff}}$ is measured with respect to the [100] direction. Especially in [100] direction ($\varphi = 0^\circ$), equation (S3) can be simplified to the expression:

\[ \varphi = 0^\circ // [100] : \theta_{\text{eff}} = \arctan \left( -\frac{\alpha}{\beta} \right). \quad (S4) \]

This establishes a connection between the $\alpha/\beta$ ratio and the angle $\theta_{\text{eff}}$, which can be detected. This allows for a quantitative estimation of $\alpha/\beta$. As stated in the main text, measurement of weak localisation (WL) in quasi-one dimensional quantum wire with in-plane magnetic field ($B_{\text{in}}$) makes it possible to detect $\theta_{\text{eff}}$, which leads to the evaluation of $\alpha/\beta$.
In this paper, to evaluate $\alpha/\beta$ for different gate voltages ($V_g$) in the vicinity of the persistent spin helix symmetry, we conduct numerical calculations for different $\alpha/\beta$ ratios. In addition to the [100] wire orientation, we model also [110] and [-110] wires for comparison.

The corresponding characteristic values of $\theta_{\text{eff}}$ are determined by equation (S3):

\[
\varphi = 45^\circ // [110]: \theta_{\text{eff}} = 135^\circ, \quad (S5)
\]
\[
\varphi = 135^\circ // [-110]: \theta_{\text{eff}} = 45^\circ. \quad (S6)
\]

Using an optimized recursive Green’s functions method, we calculated the total quantum transmission probability $T(E_F)$ at Fermi energy $E_F$, which yields the conductance in linear response within the Landauer approach: $G = G_0 T(E_F)$. For details, see Refs. 36 and 38. We use the following single particle Hamiltonian to model transport in a disordered quantum wire:

\[
H_{\text{QID}} = \frac{p_x^2 + p_y^2}{2m^*} + U_{\text{conf}}(y) + U_{\text{dis}}(x, y) + \frac{\mu_B g}{2} \left[ B_{\text{eff}}(p_y = 0) + B_{\text{in}} \right] \sigma. \quad (S7)
\]

In equation (S7), the in plane magnetic field contributing to the Zeeman interaction is given by

\[
B_{\text{in}} = B_{\text{in}} \left[ \cos(\theta_{\text{in}} - \varphi)\hat{x} + \sin(\theta_{\text{in}} - \varphi)\hat{y} \right]. \quad (S8)
\]

where $\theta_{\text{in}}$ is the angle between the magnetic field and the [100] direction. The effective magnetic field of the spin-orbit interaction is given by

\[
B_{\text{eff}} = \frac{2}{\mu_B g \hbar} \left\{ [\hat{e}_x \beta \cos 2\varphi - \hat{e}_y (\alpha + \beta \sin 2\varphi)] p_x + [\hat{e}_x (\alpha - \beta \sin 2\varphi) - \hat{e}_y \beta \cos 2\varphi] p_y \right\}. \quad (S9)
\]

The product of above expression with the vector of the Pauli matrices is, up to constant factors, given by the superposition of $H_R$ and $H_D$. Moreover, $U_{\text{conf}}(y)$ in equation (S7) represents the hard wall confining potential for the quantum wire. $U_{\text{dis}}(x, y)$ is the disorder potential, which is implemented as Anderson disorder, following a uniform random distribution. In a previous study, the occurrence of a maximum of the WL amplitude for $B_{\text{in}} \parallel B_{\text{eff}}$ is explained theoretically within the random matrix theory (RMT) framework. According to this theory,
the size of the WL amplitude depends on the symmetry class attributed to the system. In this calculation, since \( B_{\text{eff}} \) is independent of \( p_y \) and the direction of \( B_{\text{eff}} \) is fixed, \( B_{\text{eff}} (p_y = 0) \), the system possesses U(1) spin rotation symmetry, implying conservation of spin. With an additional magnetic field \( B_{\text{in}} \), usually, the U(1) spin rotation symmetry is broken for a total magnetic fields \( B_{\text{total}} \) which points into different directions, depending on the sign of \( p_y \). Consequently, spin relaxation is enhanced and WL is suppressed. However, in the special case of \( B_{\text{in}} \parallel B_{\text{eff}} \), the Hamiltonian can be written in block diagonal form:

\[
H_{\text{QID}} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad (S10)
\]

where

\[
H_\pm = \frac{p_x^2 + p_y^2}{2m^*} + U_{\text{conf}}(y) + U_{\text{dis}}(x, y) \pm \frac{\mu_B g_B}{2} B_{\text{in}} \pm \frac{1}{\hbar} \kappa' p_x, \quad (S11)
\]

with \( \kappa' = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \sin 2\varphi} \). Using \( \widetilde{H}_\pm = U_{\text{eff} \pm}^{-1} H_\pm U_{\text{eff} \pm} \) and \( U_{\text{eff} \pm} = \exp\left( \mp im^* \kappa' x / \hbar^2 \right) \), in the case of parallel effective and in-plane magnetic field, the commutation relation \([\widetilde{H}_\pm, \hat{C}] = 0\) is fulfilled, where \( \hat{C} \) is the operator of complex conjugation. This implies time reversal symmetry of \( H_- \) and \( H_+ \). Therefore, when \( B_{\text{in}} \parallel B_{\text{eff}} \), the system is described by the orthogonal symmetry class and possesses spin rotation symmetry. This leads to a suppression of spin relaxation and maximum WL amplitude. The second minor peak is attributed to a previously not considered symmetry class in the random matrix theory based approach described in Ref. 37. In the following we exclusively consider the major peak of the WL-correction, which occurs for \( B_{\text{in}} \parallel B_{\text{eff}} \).

In our numerical simulation, we used the following parameters. Phase coherence length \( L_\varphi = 1500 \) nm, spin precession length \( L_{\text{so}} = \hbar^2 / 2m^* \alpha = 360 \) nm, mean free path \( L_{\text{el}} = 202 \) nm, wire width \( W = 20 \) nm and \( E_F = 21 \) meV. In Fig. 2, we present the results of the numerical calculation for the average conductance differences \( \Delta G \) (weak localisation amplitude) as a function of \( \theta_{\text{in}} \) in...
[100], [110] and [-110] orientation for $\alpha\beta = 1/3, 2/3, 1$. Major peaks indicate the largest WL amplitude where $\theta_{\alpha}$ corresponds to $\theta_{\text{eff}}$. As anticipated, the major peak position ($\theta_{\text{eff}}$) is modulated in [100] direction, while it is fixed in [110] and [-110] directions. Moreover, in the persistent spin helix symmetry ($\alpha\beta = 1$), not only the major peak position ($\theta_{\text{eff}}$) points at $135^\circ (= \arctan(-1))$ but also the quench of WL anisotropy occurs in [-110] direction, which means the annihilation of the same strengths of $\alpha$ and $\beta$ in [-110] direction due to the antiparallel orientations.
II. WL anisotropy depending on $B_{\text{in}}$ strength

According to the theory, our proposed detection technique of $\alpha/\beta$ cannot be correctly applicable when $|B_{\text{in}}| >> |B_{\text{eff}}|$ since $B_{\text{total}}$ gets strongly aligned in the direction of $B_{\text{in}}$ and spin dephasing is equally suppressed for any $\theta_{\text{in}}$. In the present experiment, the applied in-plane field is used by constant value of $|B_{\text{in}}| = 1$ T. To confirm the validity in our measurement, we measure the $|B_{\text{in}}|$ dependence of WL anisotropy by using different strengths of $B_{\text{in}}$ (0.2 to 1 T) at different gate voltages $V_g$. The WL anisotropy in different crystal directions are plotted in Supplementary Fig. 1a-i. The detected angles $\theta_{\text{eff}}$ are almost fixed in different $|B_{\text{in}}|$ in all wire directions, which indicates the validity of the $\alpha/\beta$ evaluation in our measurement. However, for lower $|B_{\text{in}}|$, a slight shift of peak position is not so clear compared to the detected $\theta_{\text{eff}}$ in higher $|B_{\text{in}}|$ fields. This can be interpreted in the following way. For smaller $|B_{\text{in}}|$, such as $|B_{\text{in}}| = 0.2$ T, $B_{\text{in}}$ is not sufficiently strong to generate the anisotropy of the WL amplitude under variation of $\theta_{\text{in}}$, which potentially induces a slight ambiguity in the detection of the angle $\theta_{\text{eff}}$. However, $\theta_{\text{eff}}$ is fixed with sufficiently large in-plane field $|B_{\text{in}}| > 0.4$ T. We confirm the validity of our measurements for different gate voltages by the same method.

For the persistent spin helix symmetry in the [-110] direction, our numerical result shows that the WL anisotropy is quenched as $B_{\text{in}}$ becomes large compared to an annihilated $B_{\text{eff}}$ ($|B_R| - |B_D|$). The dominant $B_{\text{in}}$ generates a homogeneous dephasing rate and WL amplitudes regardless of $\theta_{\text{in}}$. However, in our experiment, a slight anisotropy of WL remains when $B_{\text{in}} = 0.6$ T (Supplementary Fig. 1i). We attribute this to the fact that $B_R$ and $B_D$ are not completely compensated by each other since $\alpha/\beta$ is evaluated as $1.07 \pm 0.04$ in the [100] wire, which represents a minor deviation from $\alpha/\beta = 1$. Also, the quench of the WL anisotropy at $B_{\text{in}} = 0.2$ T can be inferred from the fact that a magnetic field of 0.2 T is not large enough to produce WL anisotropy. The WL anisotropy of $B_{\text{in}} = 0.6$T at $V_g = -9$V in the [-110] wire is still much smaller.
compared to that at other values of $V_g$ i.e. WL anisotropy at $B_{in}$ = 0.6 T at $V_g$ = -5V in Supplementary Fig. 1f. Considering these points, the quench of the anisotropy in our measurement can be regarded as a fingerprint of the persistent spin helix symmetry to support the result that we find $\alpha/\beta = -\tan(\theta_{eff} = 133^\circ) = 1.07 \pm 0.04$ in [100] the wire.

Supplementary Fig. 1: $B_{in}$ dependence of the WL anisotropy. a-h, WL anisotropy (WL amplitude $\Delta \sigma'$ as a function of $\theta_{in}$) depending on the strength of $B_{in}$ at $V_g$ = 0V, -5V and -9V in the [110], [100] and [-110] wires respectively, except for $V_g$ = -9V in the [-110] wire. We expose the sample to $|B_{in}|$ with magnitudes ranging from 0.2 to 1 T for each wire orientation. i. The WL anisotropy depending on the strength of $B_{in}$ at $V_g$ = -9V in [-110] direction. The offset magneto-conductance is added to the measured values.
III. Calculations for the evaluation of the absolute values of SOI parameters

It is suggested that the WL amplitude exhibits a minimum when $|B_{\text{in}}|$ is very close to $|B_{\text{eff}}|$ \cite{31,37}. This is because spin relaxation is caused by randomization of $B_{\text{total}}$ which depends on electron momentum. For example, when $|B_{\text{in}}|$ is much larger than $|B_{\text{eff}}|$, strong alignment of $B_{\text{total}}$ and $B_{\text{in}}$ suppresses spin relaxation, and when $|B_{\text{in}}|$ is much smaller than $|B_{\text{eff}}|$, the full WL amplitude is recovered. Therefore, the amplitude of WL is modulated by changing the ratio $|B_{\text{in}}|/|B_{\text{eff}}|$. From above simple arguments, we propose a technique to determine the absolute values of Rashba and Dresselhaus SOI parameters ($\alpha$ and $\beta$) as well as $|B_{\text{eff}}|$.

This technique is based on the idea that spin relaxation is maximized when $B_{\text{in}}$ is very close to $B_{\text{eff}}$. Therefore, we first confirm this concept within a qualitative calculation based on a toy model. In the toy model, we calculate the probability of spin preservation for an electron spin which experiences spin precession during forward and backward propagation: denoted as clockwise (CW) and counterclockwise (CCW) paths. The schematic image of this model is shown in Supplementary Fig. 2a. The vectors of $B_{\text{total1}}$ and $B_{\text{total2}}$ in the figure are defined by $B_{\text{total1}} = B_{\text{eff}}(+p) + B_{\text{in}}$ and $B_{\text{total2}} = B_{\text{eff}}(-p) + B_{\text{in}}$ respectively. Here, we define $\theta$ as the angle between $B_{\text{in}}$ and $B_{\text{eff}}$. $\theta_1$ and $\theta_2$ are angles of $B_{\text{total1}}$ to $B_{\text{eff}}(+p)$ and $B_{\text{total2}}$ to $B_{\text{eff}}(-p)$ (see Supplementary Fig. 2a). To conduct the calculation, we define the spin rotation operators as

$$R_n (\phi) = I \cos(\phi/2) - i(\sigma \cdot \bar{m})\sin(\phi/2), \quad (S11)$$

$$R_n (\phi) = I \cos(\phi/2) - i(\sigma \cdot \bar{n})\sin(\phi/2), \quad (S12)$$

where $I$ is a unit matrix and $\phi$ are spin precession angles around the axis of $B_{\text{total1}}$ and $B_{\text{total2}}$ respectively. $\bar{m} = (\cos \theta_1, \sin \theta_1, 0)$ and $\bar{n} = (-\cos \theta_2, \sin \theta_2, 0)$ are unit vectors of $B_{\text{total1}}$ and $B_{\text{total2}}$ respectively. The final spin states along CW and CCW paths are given by the following expressions:
\[ |f_{CW}\rangle = R_{m_1} (\phi_1) R_{m_2} (\phi_2) R_{m_3} (\phi_3) |i_{\text{spin}}\rangle, \quad (S13) \]

\[ |f_{CCW}\rangle = R_{m_1} (\phi_1') R_{m_2} (\phi_2') R_{m_3} (\phi_3') |i_{\text{spin}}\rangle. \quad (S14) \]

where \( |i_{\text{spin}}\rangle \) is the initial spin state and we set \( |i_{\text{spin}}\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) for convenience. The probability of spin preservation \( P \) in the interference at the original point is described as

\[ P = \left( |f_{CW}\rangle + |f_{CCW}\rangle \right)^* \left( |f_{CCW}\rangle + |f_{CW}\rangle \right). \quad (S15) \]

Since \( \theta_1, \theta_2, B_{\text{total}1} \) and \( B_{\text{total}2} \) are related to \( B_{\text{in}}, B_{\text{eff}} \) and \( \theta \), the spin preservation probability \( P \) is regarded as a function of \( B_{\text{in}}, B_{\text{eff}} \) and \( \theta \). Here, we fix \( \|B_{\text{eff}}\| = 1 \) and calculate the probability \( P \) by varying \( B_{\text{in}} \) for several different angles \( \theta \). The calculated results are shown in Supplementary Fig. 2b. When \( \theta = 0^\circ, 180^\circ \), the spin preservation probability is fixed at unity, which indicates that no additional spin relaxation is induced by \( B_{\text{in}} \) regardless of the magnetic field strengths. This can be explained by the fact that spin relaxation is suppressed for unidirectional \( B_{\text{total}} \). In contrast, for the case of other angles such as \( \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ \), the spin preservation probability changes as a function of the magnitude of \( B_{\text{in}} \) and gets minimal when \( |B_{\text{in}}| \approx |B_{\text{eff}}| = 1 \), as expected. Thus, we confirm that the spin relaxation rate is maximized when the external magnetic field \( B_{\text{in}} \) is comparable with \( B_{\text{eff}} \) except for \( \theta = 0^\circ, 180^\circ \) (\( B_{\text{in}} \parallel B_{\text{eff}} \)).

Above simplified model is one-dimensional, in contrast to the numerical results of Fig. 6 a). If we plot the WL amplitude with respect to \( |B_{\text{in}}| / |B_{\text{eff}}| \), we observe a minimum in the WL amplitude difference at the same positions as for the other orientations of the in-plane field. This observation, depicted in Supplementary Fig. 3, is due to the fact that the numerical model is describing a two-dimensional structure with impurity scattering, which changes momentum direction and therefore allows changes of the spin precession axis orientation. This finding is expected to be observable in narrow quantum wires with small Fermi energies, in contrast to the energy scales
used in the experiment.

Supplementary Fig. 2: Inference of spin relaxation from the toy model. a. Schematic images of the modeled processes. In the quasi-one dimensional quantum wire the effective magnetic field $B_{\text{eff}}$ is unidirectional regardless of the electron momentum. $B_{\text{total1}}$ and $B_{\text{total2}}$ are defined by a superposition of $B_{\text{eff}}$ and the magnetic in-plane field $B_{\text{in}}$, resulting in total fields of $B_{\text{total1}} = B_{\text{eff}}(+p) + B_{\text{in}}$ and $B_{\text{total2}} = B_{\text{eff}}(-p) + B_{\text{in}}$ respectively. In the reference frame of the moving electron spin precesses in the two paths associated with clockwise (CW) and counterclockwise (CCW) spin rotation respectively. Both paths interfere at the point of origin. 

b. Dependence of the resulting spin preservation probability $P$ due to the interference in different direction $\theta$ (the angle between $B_{\text{in}}$ and $B_{\text{eff}}$). We fix $|B_{\text{eff}}| = 1$ here. The shown values of $\theta$ include $\theta = 0^\circ$, $180^\circ$ (green rectangular), $\theta = 30^\circ$, $150^\circ$ (red circle) and $\theta = 60^\circ$, $120^\circ$ (blue triangle). Lower probability of spin preservation indicates larger spin relaxation, induced by precession around the axis defined by $B_{\text{total}}$. An increase of spin relaxation gives rise to a diminished WL signal.
Supplementary Fig. 3: Amplitude difference of the weak-localisation correction. Data shown for in-plane field orientation of $\theta = 0^\circ$ ($B_{\text{in}} \parallel B_{\text{eff}}$) in the numerical model analyzed in Fig. 6 a). In contrast to the toy model, a dip at equal field strength is observed. This is explained by the finite transversal extent of the wire and the presence of disorder scattering.

Since the WL amplitude in in-plane field is suppressed together with the spin relaxation processes, we anticipate that the WL amplitude should be most suppressed when $|B_{\text{in}}|/|B_{\text{eff}}|$. In order to confirm this point we further conduct numerical simulations using the recursive Green’s function scheme introduced in paragraph 1. The $|B_{\text{in}}|/|B_{\text{eff}}|$ dependence of the WL amplitude is investigated in [100] quasi-one dimensional wires. In this simulation, we choose the following numerical parameters: Phase coherent length $\phi_L = 1000$ nm, spin precession length $L_{\alpha} = \alpha^2 / 2m^* \alpha = 166$ nm, mean free path $L_{\text{el}} = 17$ nm and wire width $W = 100$ nm. As the result shown in Fig. 6a indicates, minimum amplitudes of WL appear when the $|B_{\text{in}}|/|B_{\text{eff}}|$ ratio is around 1.0 for different $\theta$ angles ($\theta = 60^\circ$, $120^\circ$, $150^\circ$). The deviation of the curves recorded for $\theta = 30^\circ$ is in line with the fact that large number of propagating orbital channels is not supported in the numerical scheme. As a consequence of this, non-universal effects on the
individual mode cannot be negligible in calculation, which explain the observed deviation. Hence, it is possible to estimate the approximate $B_{\text{eff}}$ strength by using the relation of $|B_{\text{in}}|/|B_{\text{eff}}| = 1$, for which we observe the minimum WL amplitude.

Moreover, when we convert the deduced $|B_{\text{eff}}|$ (the unit is tesla) in [100] wire into the absolute values of $\alpha$ and $\beta$ (the unit is eV·m), we use the following equation,

$$|B_{\text{eff}}| = 2\sqrt{\alpha^2 + \beta^2} k_F/g\mu_B, \quad (S16)$$

Here, $g$ is the effective Landé g-factor and we assume $g = 3.5$ (Ref. 40). $\mu_B$ is the Bohr magneton and $k_F$ is the Fermi wave number in 2DEG. By applying equation (S16) together with the estimated $|B_{\text{eff}}|$ and the $\alpha/\beta$ ratio obtained from WL anisotropy measurements in the [100] wire, $\alpha$ and $\beta$ can be deduced.
Supplementary References


40. Nitta, J., Lin, Y., Akazaki, T. & Koga, T. Gate-controlled electron $g$ factor in an InAs-inserted-channel In$_{0.53}$Ga$_{0.47}$As/In$_{0.52}$Al$_{0.48}$As heterostructure. *Appl. Phys. Lett.* **83**, 4565 - 4567 (2003).