Supplementary 1: Size and stability of an isolated skyrmion varying different parameters

We have studied the size of the isolated skyrmion as a function of the dot diameter for different values of \( D \) (see Fig. S1A). At low \( D \) values, their size almost does not depend on the diameter whereas for larger \( D \), the total energy is lowered by increasing the number of twisted spin pairs leading to an increase of the size of the skyrmion up to dimensions that are limited by the dot edges. Interestingly, we see that the skyrmion size can be controlled by decreasing the nanodisk diameter, leading to very small skyrmions even for very large \( D \) values.

In Fig. S1B, we present the evolution of the skyrmion size as a function of an external perpendicular field \( H_z \) by making simulations in a 80 nm wide nanodisk with a single skyrmion in the relaxed state. Starting from zero field, we applied a magnetic field of increasing intensity until the skyrmion was annihilated and the system returned to the FM state. These simulations were repeated for both external field directions and for various values of \( D \) and \( K \). Firstly, we observed that the skyrmion radius varied with the applied field, decreasing for a field anti-parallel to its central magnetization and increasing for the opposite orientation. The annihilation mechanism under field is then fundamentally different in the two cases. With an antiparallel field, the skyrmion is contracted and finally is annihilated when the Zeeman energy term overcomes the exchange term responsible for the topological stability of the skyrmion. With a parallel \( H_z \) field, on the other hand, the skyrmion expands beyond the borders of the disk, in a topologically allowed path, needing only to overcome the opposite tilting border effect described earlier. Finally we notice that the total energy of the system is not significantly modified under the application of an in-plane field, and the field induces only a small and reversible distortion of the skyrmion shape.

In Fig. S2, we display the parameter regions where the FM state is stable (in yellow), the isolated skyrmion is stable (in blue), and where both these states are stable (in green) as a function of DMI parameter \( D \) for varying uniaxial anisotropy \( K \) (Fig. S2a.), varying exchange stiffness \( A \) (Fig. S2b.), and varying perpendicular field \( H_z \) (Fig. S2c.)
Supplementary 2: Nucleation mechanism in presence of a large DMI

In order to get insights on the skyrmion nucleation mechanism, we present in Fig S3 a series of snapshots of both the magnetization distribution along z direction and the DMI energy density in a 80 nm disk obtained at different times after the injection of a pulse of spin polarized current. The initial state at t=0 was uniformly magnetized along +z (red). The pulse starts at t = 0 and has a duration of 2ns long pulse. The spin-polarized current (P = 0.4 oriented along -z) is injected, as for the simulations presented in Fig.2, in the central region of the disk (40nm). For these simulations, the D value is D = 6 mJ/m², the current density is J = 2x10⁸ A/cm² and the magnetic damping is α = 0.3.

At t = 0.21 ns, the magnetization is still quasi uniform and oriented along +z with only a small tilt induced on the disk edge by the DMI (independently of the current)⁵¹. The blue contrast on the edge of the right figure reflects the lowering of the DMI energy by this inward tilt. The initial relaxed quasi uniformly magnetized state remains almost unchanged during the first 0.2 ns. Note that the duration of this initial waiting period is strongly dependent on small perturbations of the initial magnetization distribution and can be drastically reduced for example by introducing a small tilt from the vertical in the direction of the spin polarization. At t = 0.27 ns, the magnetization inside a region under the injection contact region is dynamically reversed (central part is along –z) and this configuration corresponds to a magnetic bubble with a skyrmion number that remains close to zero i.e. equivalent topologically to the initial uniform state⁵². The blue and red regions in the energy distribution correspond to different chiralities, decreasing or increasing the DMI energy, in the pseudo-wall surrounding the bubble. Then the spins inside the pseudo-wall of the bubble tend to be oriented very rapidly (t = 0.29 ns) toward the direction that is favored by the DM interaction, compressing the portion of non-favored chirality into a small region (indicated by the yellow arrow) that concentrates a large amount of exchange and DM energy. The size of this small defect-like region, comparable to a Bloch point in classical bubble material, decreases with time and changes its position due to the spin transfer induced dynamics of the bubble. Indeed this process resembles to what is observed in garnet materials without DM interaction in which the wall of this bubble contains two Bloch lines wall portions of different chiralities (the information unit of the so-called Bloch line memory⁵³). In our system, the influence of DMI is to break the symmetry between the two possible chiralities. At t = 0.31 ns, the defect size becomes small enough and eventually switches its configuration (with emission of spin waves across the disk) with an increase of the skyrmion number close to 1 (not exactly 1 because of the tilted magnetization on the edges) that corresponds to a single skyrmion. Indeed such mechanism of nucleation has been already identified in the Bloch line memory physics, where also the topological barrier has to be overcome in order to write and erase information⁵⁴. In the thick bubble materials, it involves the injection and travel across the sample thickness of a Bloch
point (the same process was also invoked for the reversal of a vortex core in NiFe microdisks\textsuperscript{25}). The only difference in the present situation of an ultrathin magnetic film is that the notion of a Bloch point becomes meaningless at a thickness of two atomic planes. If we go to the single atomic layer limit, the "injection and crossing" of a Bloch point simply amount to the collective switching of typically three neighboring atomic moments from plus to minus orientation (in the plane), that subsequently spreads along the domain wall and across it. Once the skyrmion configuration is nucleated, this relaxed configuration remains stable during the remaining time of the current pulse and after the end of the pulse (see also Supplementary Video).

By changing the disk shape and the material parameters, we have found that in some cases that the small defect region reaches the disk edges due to the attractive interaction with the tilted magnetization, leading to the transformation of the bubble into an edge-to-edge domain wall. Finally, in other conditions, we have also observed some more complex nucleation process involved the nucleation of more than one skyrmion (sometimes together with edge-to-edge domain wall). However they eventually collapse before the end of the pulse in order to relax in a final configuration corresponding to a single skyrmion in the disk.
Supplementary 3: Simulations with smaller cells for the small D limit

At the lowest values of $D$ the skyrmions reach the size of a few simulation discretization cells (at $D = 2.5 \text{ mJ/m}^2$, the skyrmion diameter is $8 \text{ nm} = 8$ cells). It is then necessary to reduce the discretization cell size, which lowers the threshold value of $D$ for the existence of a meta-stable skyrmion in the dot. With a cell size of the order of the inter-atomic length ($0.2 \text{ nm}$), meta-stable skyrmions can be found down to $D = 1.9 \text{ mJ/m}^2$. Small skyrmion sizes are also be obtained by shrinking the disk diameter (see inset of fig 1), leading to very small skyrmions even for very large $D$ values.

Supplementary 4: Thiele equation for an isolated skyrmion

The motion of an isolated rigid skyrmion in a track of thickness $h$ and cross-section $S$ without spin transfer torques can be analytically described with the formalism introduced by Thiele using the equations:

\[
F + G \times v + \alpha D v = 0
\]

\[
G \equiv -\frac{M_S}{\gamma} \int \sin \theta \nabla \theta \times \nabla \phi \, d^3r = \pm \frac{M_S h}{\gamma} 4\pi e_z
\]

\[
D_{ij} \equiv -\frac{M_S}{\gamma} \int \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} + \sin^2 \theta \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \, d^3r = D \delta_{ij}
\]

\[
D_{xx} = -\frac{M_S h}{\gamma} \int \left( \frac{\partial m}{\partial x} \right)^2 \, dx dy = -\frac{M_S h 2S}{\gamma \Delta T}
\]

where $v$ is the skyrmion velocity, $F$ represents forces exerted on the skyrmion (e.g. by the track borders), $G$ is called the gyrovector (the sign depending on the skyrmion charge sign), $D$ is the dissipation matrix, $\Delta T$ is the Thiele length and $e_x, e_y, e_z$ are the unit vectors along the three spatial axis.

The pinning due to the track borders can be introduced in a first approximation as $F = -\kappa y e_y$.

Also in the case of a track, the velocity in the steady state is along the axis of the track: $v = v e_x$.

Note that recent numerical and analytical studies have shown that some features of the dynamics of magnetic bubbles having a non zero skyrmion number, deviate from this standard Thiele approach and that a better description is obtained if the mass of the magnetic bubble is taken into account.

Here we limit our description to the case of massless skyrmions and therefore do not describe effects related to such inertial dynamics.

Under a CIP spin transfer torque, the Thiele equation is modified to

\[
F_{\text{CIP}} = -\kappa_y \cdot \theta \cdot e_y
\]
\[ F + G \times (v - u) + D(\alpha v - \beta u) = 0 \]

Where \( u \) is the normalized current introduced before. This yields for the skyrmion in a track:

\[ v = \frac{\beta}{\alpha} u \quad \text{and} \quad y = \frac{G}{\kappa}(v - u) = \frac{G}{\kappa}\alpha(\beta - \alpha) \]

which is consistent with the results of fig. 3. Notice that for the skyrmion in an infinite film with low damping \( v_\parallel \approx \alpha \) and different from zero. As \( v_\perp \) is not zero introduces a force term independent of \( \beta \) in the Thiele equation, the skyrmion lattice moves even for \( \beta = 0 \), in contrast to our situation of skyrmions in a track.\(^{510}\)

For the CPP geometry with current polarization \( P \), considering only the Slonczewski torque, the Thiele equation becomes\(^{511}\)

\[ F + F_{\text{STT}} + G \times v + \alpha Dv = 0, \quad F_{\text{STT}} = \pm \frac{1}{2} e B \pi b e_z \times P = f_{\text{STT}} e_z \times e_p \]

Where \( b \) is a characteristic length of the skyrmion, \( J \) the current density, and the sign of \( F_{\text{STT}} \) depends on the skyrmion charge sign. Solving for case of the track as above for a polarization \( P = P e_z \):

\[ v = -\frac{f_{\text{STT}}}{u} \quad \text{and} \quad y = \frac{G}{\kappa} v. \]

For the case \( P = P e_z, v = y = 0 \), and for the case \( P = P e_x, v = 0 \) and \( y = -f_{\text{STT}}/\kappa \).

References in Supplementary


Figure captions in Supplementary

**Figure S1** | Skyrmion diameter as a function of $D$ for different disk diameters (a) and of external field $Hz$ for different $D$ values (b).

**Figure S2** | Interval of skyrmion stability, for (a.) different values of $D$ and $K$ ($Hz=0$), (b.) $D$ and $Hz$ ($K=0.8$ MJ/m$^3$), and (c.) $D$ and exchange stiffness. The yellow shading represents region of stability of the FM state, the blue shading the stability of the isolated skyrmion, and the green shading the region of bi-stability.

**Figure S3** | Magnetization distribution along $z$ (left) and DMI energy density distribution (right) in 80 nm nanodisks at different times ($t = 0.21$ ns, 0.27 ns, 0.29 ns, 0.31 ns, 0.32 ns and 0.38 ns) during a 2ns long pulse of spin-polarized current ($P = 0.4$ along -z) injected in the central region of the disk (40nm). The $D$ value is $D = 6$ mJ/m$^2$; $J = 2\times10^8$ A/cm$^2$ and $\alpha = 0.3$. The initial state at $t=0$ was quasi-uniformly magnetized along $+z$ (red). The yellow arrows at $t = 0.29$ and 0.31 ns indicate the location of defect-like point inside the wall of the bubble having a strong positive DMI energy density. After the reversal of the spins in the defect (at $t = 0.32$ ns for these parameter conditions), a skyrmion is stabilized at the center of the disk and remains stable even after the end of the current pulse.
SUPPLEMENTARY

Figure S1 Size of isolated skyrmions

![Graph a. Skyrmion diameter vs. D (mJ/m²)]

![Graph b. Skyrmion diameter vs. Magnetic field B_Z (mT)]
SUPPLEMENTARY

Figure S2: Stability diagram of skyrmions

a. stable FM (no skyrmion)
b. Bi-stable region
   stable isolated skyrmion
c. Bi-stable region
   stable isolated skyrmion
Supplementary
Figure S3 | Example of nucleation mechanism