Supplement A: Plane Wave Expansion Method and the Low-Energy Hamiltonian

In a bi-anisotropic medium with the following form of constitutive parameters,

\[
\varepsilon = \begin{pmatrix}
\epsilon_\perp & 0 & 0 \\
0 & \epsilon_\perp & 0 \\
0 & 0 & \epsilon_{zz}
\end{pmatrix}, \quad \mu = \begin{pmatrix}
\mu_\perp & 0 & 0 \\
0 & \mu_\perp & 0 \\
0 & 0 & \mu_{zz}
\end{pmatrix}, \quad \chi = \begin{pmatrix}
0 & i\chi_{xy} & 0 \\
i\chi_{yx} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (1SA)
\]

\(D\) and \(B\) are related to \(E\) and \(H\) by the relations \(D = \varepsilon E + \chi H\) and \(B = \mu E + \chi H\) where by reciprocity \(\chi = \chi^\dagger\) with \(\dagger\) being the conjugate transpose operator. Note that we have assumed the bi-anisotropic parameters \(\chi_{xy}\) and \(\chi_{yx}\) are purely real quantities implying a non-lossy medium. In a triangular lattice of circular rods made of such a medium in the absence of the bi-anisotropy \((\chi_{xy} = \chi_{yx} = 0)\) there are two sets of Dirac cones at each \(K\) and \(K'\) point corresponding to both TE \((E_z = 0, H_z \neq 0)\) and TM \((E_z \neq 0, H_z = 0)\) polarizations (see Fig. 1SA). We found that a gap opens at the Dirac point only for the case of a medium with \(\chi_{xy} \neq \chi_{yx}\). Here for simplicity we assume an asymmetric form of bi-anisotropy \(\chi_{xy} = -\chi_{yx}\) which also allows maintaining isotropic dispersion near \(K\) and \(K'\) points after removal of the Dirac point’s degeneracy.

From the Maxwell’s equations, one can express the tangential components of the fields in terms of their \(z\) components and finally obtain a reduced set of equations for the \(z\) components of the fields:

\[
\begin{align*}
\left[k_0^2\mu_{zz} + \partial_x m \partial_x + \partial_y m \partial_y\right] H_z &= -i\left[\partial_x \theta \partial_y - \partial_y \theta \partial_x\right] E_z, \\
\left[k_0^2\epsilon_{zz} + \partial_x e \partial_x + \partial_y e \partial_y\right] E_z &= -i\left[\partial_x \theta \partial_y - \partial_y \theta \partial_x\right] H_z,
\end{align*}
\]  

(2SA)

where \(m = \mu_\perp/ (\mu_\perp \epsilon_\perp - (\chi_{xy})^2)\), \(e = \epsilon_\perp/ (\mu_\perp \epsilon_\perp - (\chi_{xy})^2)\), and \(\theta = \chi_{xy}/ (\mu_\perp \epsilon_\perp - (\chi_{xy})^2)\). We also assume \(\epsilon_{zz} = \mu_{zz}\), and \(\epsilon_\perp = \mu_\perp\), as explained in the main text. In this case \(e = m\) and one can introduce two uncoupled polarizations, \(\psi^+ = E_z + H_z\) and \(\psi^- = E_z - H_z\) that satisfy the relation \(\mathcal{L}_0 \psi^\pm = \pm \mathcal{L}_1 \psi^\pm\), where \(\mathcal{L}_0 = k_0^2 \epsilon_{zz} + \partial_x e \partial_x + \partial_y e \partial_y\) and \(\mathcal{L}_1 = -i\left[\partial_x \theta \partial_y - \partial_y \theta \partial_x\right]\). These two polarizations constitute the basis for photon spin. The eigenvector with positive eigenvalue \((\psi^+)\) plays the role of spin-up and the other one \((\psi^-)\) with negative eigenvalue plays the role of spin-down photonic states.

To find solutions of Eq. (2SA) for the case of periodic meta-crystal we utilize the periodicity of the structure, and expand the fields and the constitutive parameters in terms of plane waves,
\[ E_z(r; q) = \sum G \cdot E_G e^{i(q+G) \cdot r}, \quad H_z(r; q) = \sum G \cdot H_G e^{i(q+G) \cdot r}, \]
\[ e(r) = \sum G \cdot e_G e^{iG \cdot r}, \quad \theta(r) = \sum G \cdot \theta_G e^{iG \cdot r}, \]  
(3SA)

Spatially constant perpendicular components for the permittivity and the permeability are assumed. By substituting this form of the fields and the constitutive parameters into Eq. (2SA), we obtain a system of linear equations for the Fourier components of the fields,

\[ k_0^2 \varepsilon_{zz} H_G - \sum G' \cdot e_{G-G'} \cdot \left\{ (q_x + G_x)(q_x + G_x') + (q_y + G_y)(q_y + G_y') \right\} H_{G'} = -i \sum G' \cdot \theta_{G-G'} \cdot \left\{ (q_x + G_x)(q_y + G_y') - (q_y + G_y)(q_x + G_x') \right\} E_{G'} \]
\[ k_0^2 \varepsilon_{zz} E_G - \sum G' \cdot e_{G-G'} \cdot \left\{ (q_x + G_x)(q_x + G_x') + (q_y + G_y)(q_y + G_y') \right\} E_{G'} = i \sum G' \cdot \theta_{G-G'} \cdot \left\{ (q_x + G_x)(q_y + G_y') - (q_y + G_y)(q_x + G_x') \right\} H_{G'} \]  
(4SA)

We then find the eigenfrequencies and eigenmodes of the meta-crystal by numerically solving this system of linear equations for a sufficiently large number of plane waves so that the convergence is achieved.

To describe modes near the band crossing we focus on the vicinity of the K and K’ points. Let us first consider the K point. In this case we truncate the plane wave basis to only the plane waves corresponding to the three nearby Gamma points. These plane waves correspond to the three equal-length reciprocal vectors \( K + G_i \) each rotated 120° with respect to one another, where \( K = [K, 0, 0] \) \((K = 4\pi/3a)\) is a vector pointing from the \( \Gamma \) point to the K point, and \( i = 0, 1, 2 \) indexes the three reciprocal lattice vectors. Then a 6x6 Hermitian equation can be obtained for the Fourier components of the fields:

\[ \begin{pmatrix} \hat{\varepsilon} & \hat{\theta} \\ \hat{\theta}^\dagger & \hat{\mu} \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = k_0^2 \varepsilon_{zz} \begin{pmatrix} E \\ H \end{pmatrix} \]
(5SA)

where \( E = [E_{G_0}; E_{G_1}; E_{G_2}] \), and \( H = [H_{G_0}; H_{G_1}; H_{G_2}] \).
Fig. 1SA: Schematic band structure of the TE- and TM-polarized modes before introduction of the bi-anisotropy. For each polarization, only the singlet and the two doublet modes (labeled by ‘1’ and ‘2’) are shown. Here a slight mismatch between permeability and permittivity is assumed for better illustration and \( \omega_{bp}^2(K) = K\left[\left(e_{g_0} + \frac{1}{2} e_{g_1}\right) / \varepsilon_{xx}\right]^{1/2} \) and \( \omega_{bM}^2(K) = K\left[\left(m_{g_0} + \frac{1}{2} m_{g_1}\right) / \mu_{xx}\right]^{1/2} \).

In the vicinity of the K point \( q_x = K + \delta k_x \) and \( q_y = \delta k_y \), and by neglecting terms of the order \( \delta k_i \delta k_j \), \( \chi \delta k_i \), and \( \chi^2 \) and higher, the matrices \( \hat{\epsilon} \), \( \hat{\mu} \), and \( \hat{\theta} \) are found to be

\[
\hat{\epsilon} = \hat{\mu} = K^2 \left( \begin{array}{ccc}
e_{g_0} & -\frac{1}{2} e_{g_1} & -\frac{1}{2} e_{g_1} \\
-\frac{1}{2} e_{g_1} & e_{g_0} & -\frac{1}{2} e_{g_1} \\
-\frac{1}{2} e_{g_1} & -\frac{1}{2} e_{g_1} & e_{g_0} \end{array} \right) + K \delta k_x \left( \begin{array}{cc}
\frac{2e_{g_0}}{1} & e_{g_1} \\
\frac{1}{2} e_{g_1} & -e_{g_0} & -e_{g_1} \\
\frac{1}{2} e_{g_1} & -e_{g_1} & -e_{g_0} \end{array} \right) + K \delta k_y \left( \begin{array}{ccc}
0 & -\frac{\sqrt{3}}{2} e_{g_1} & \frac{\sqrt{3}}{2} e_{g_1} \\
-\frac{\sqrt{3}}{2} e_{g_1} & -\sqrt{3} e_{g_0} & 0 \\
\frac{\sqrt{3}}{2} e_{g_1} & 0 & \sqrt{3} e_{g_0} \end{array} \right),
\]

\( (6SA) \)

\[
\hat{\theta} = i\frac{\sqrt{3}}{2} K^2 \theta_{1.g_1} \left( \begin{array}{ccc}0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0 \end{array} \right),
\]

In the above equations, we have assumed that all the constitutive parameters have cylindrical symmetry according to the shape of rods forming the meta-crystal. Thus the Fourier component corresponding to the reciprocal lattice vector \( G_2 \) is the same as that of \( G_1 \). Using the transformation \([\hat{G}, \hat{O}; \hat{O}, \hat{O}] \) where \( \hat{G} = 1/\sqrt{3}[1,1,1; 1, \eta^2, \eta; 1, \eta, \eta^2] \) and \( \eta = \exp(i \frac{2\pi}{3}) \), one can arrive at two (coupled) copies of Dirac bands by eliminating the two singlets and keeping the two doublets:

\[
\left( \begin{array}{c}v_D (\delta k_x \hat{\sigma}_x + \delta k_y \hat{\sigma}_y) \\ \zeta \hat{\sigma}_z \end{array} \right) = \Omega_{PTI}(\delta k) \left( \begin{array}{c}E_d \\ H_d \end{array} \right),
\]

\( (7SA) \)

where \( \Omega_{PTI}(\delta k) = \epsilon_{zz} \omega(K) \omega^2(\delta k) - \omega^2(K) e_{g_0} e_{g_1} / e_{zz} K^2 \), \( \omega(K) = Kc \left[\left(e_{g_0} + \frac{1}{2} e_{g_1}\right) / \varepsilon_{zz}\right]^{1/2} \), \( v_D = \omega(K) / K \left(e_{g_0} - e_{g_1}\right) \), \( \zeta = \frac{3\omega(K)}{2} \theta_{1,g_1}, E_d = [E_x; E_z], \) and \( H_d = [H_x; H_z] \). After diagonalization using the transformation \( \hat{O}_2 = [\hat{I}, \hat{I}; \hat{I}, -\hat{I}] \), where \( \hat{I} \) is 2 × 2 identity matrix, we arrive at a set of two massive Dirac dispersions for two spin configurations:
\[
\begin{pmatrix}
v_D (\delta k_x \sigma_x + \delta k_y \sigma_y) + \zeta \sigma_z & 0 \\
0 & v_D (\delta k_x \sigma_x + \delta k_y \sigma_y) - \zeta \sigma_z
\end{pmatrix}
\begin{pmatrix}
\psi_+^K \\
\psi_-^K
\end{pmatrix}
= \Omega_{\text{PTI}} (\delta k)
\begin{pmatrix}
\psi_+^K \\
\psi_-^K
\end{pmatrix},
\] (8SA)

where \( \psi_+^K = E_d + H_d \) and \( \psi_-^K = E_d - H_d \).

The same procedure can be repeated for the \( K' \) point for which one obtains an effective Hamiltonian,

\[
\begin{pmatrix}
0 & v_D (-\delta k_x \sigma_x + \delta k_y \sigma_y) + \zeta \sigma_z \\
v_D (-\delta k_x \sigma_x + \delta k_y \sigma_y) - \zeta \sigma_z & 0
\end{pmatrix}
\begin{pmatrix}
\psi_+^{K'} \\
\psi_-^{K'}
\end{pmatrix}
= \Omega_{\text{PTI}} (\delta k)
\begin{pmatrix}
\psi_+^{K'} \\
\psi_-^{K'}
\end{pmatrix},
\] (9SA)

By introducing valley and spin Pauli matrices \( \hat{\sigma}_i \) and \( \hat{\tau}_i \), respectively, the total 8x8 Hamiltonian (combining Eqs. (8SA) and (9SA)) can be written in a compact form:

\[
\hat{\mathcal{H}} = v_D (\hat{\sigma}_x \delta k_x + \delta k_y \hat{\sigma}_y) + \zeta \hat{\sigma}_z \hat{\sigma}_z,
\] (10SA)

which is identical to the electronic Hamiltonian introduced by Kane and Mele in Ref. [2] and given in the main text Eq. (2). Note that the matrices \( \hat{\sigma}_i \), \( \hat{\tau}_i \) and \( \hat{s}_i \) are the same Pauli matrices but acting on different subspaces. The matrices \( \hat{\tau}_i \) act on the band subspace corresponding to the two-fold Dirac degeneracy. The matrices \( \hat{\sigma}_i \) act on the valley subspace corresponding to the pair of Brillouin zone corners \( K \) and \( K' \) connected by the time-reversal symmetry. The matrices \( \hat{s}_i \) act on the polarization (spin) components of the wavefunction. Thus, the Hilbert space defined by the Hamiltonian \( \hat{\mathcal{H}} \) is an eight dimensional space spanned by its eight-component eigenfunctions.

When written for two spins separately, as given in the main text, two Hamiltonians for spin-up and spin-down take the following form:

\[
\hat{\mathcal{H}}_{\pm}(\delta k) \Psi_{\pm}(\delta k) =
\begin{pmatrix}
\pm \zeta & (\delta k_x - i \delta k_y) v_D & 0 & 0 \\
(\delta k_x + i \delta k_y) v_D & \mp \zeta & 0 & 0 \\
0 & 0 & \mp \zeta & (-\delta k_x - i \delta k_y) v_D \\
0 & 0 & (-\delta k_x + i \delta k_y) v_D & \pm \zeta
\end{pmatrix}
\]

\[
\Psi_{\pm}(\delta k) = \begin{pmatrix}
\psi_+^{K + \delta k} \\
\psi_-^{K + \delta k} \\
\psi_+^{K' - \delta k} \\
\psi_-^{K' - \delta k}
\end{pmatrix},
\] (11SA)

\[
\Omega_{\text{PTI}} (\delta k)
\begin{pmatrix}
\Psi_{\pm}(\delta k)
\end{pmatrix}
\]

where \( \Psi_{\pm}(\delta k) \equiv [ \psi_+^{K + \delta k}, \psi_+^{K - \delta k}, \psi_+^{K' - \delta k}, \psi_+^{K' - \delta k} ]^T \).
We note that according to our simulations, a slight $\epsilon\mu$ mismatch $\delta = \mu_\perp - \epsilon_\perp$ introducing mixing of the spin-up and spin-down states can be tolerated for sufficiently large $\chi_{xy}$, and robust edge states survive as long as the gap due to the bi-anisotropy exceeds the spectral detuning between the TE and TM bands caused by a finite value of $\delta$. This situation can be described by adding an additional term of the form $\Delta \hat{H} \sim \delta \hat{\tau}_z \hat{\sigma}_0 \hat{\delta}_x$ to the effective Hamiltonian given by Eq. (10SA), where $\hat{\tau}_0$ and $\hat{\sigma}_0$ are the identity matrices acting on the valley and band subspaces, respectively.

Supplement B: Matching the Frequency Gaps of Trivial and Nontrivial Photonic Insulators

In contrast to electronic systems where the topologically protected edge states can exist at the interface with vacuum, this in general cannot happen for photonic systems since electromagnetic field will leak into free space. To avoid such leakage for photonic systems one has to propose a mechanism for confinement of the field. This is usually accomplished by stacking the photonic structure of interest with another structure possessing a complete band gap in the spectrum. The hexagonal meta-crystal considered here possesses a complete photonic band gap in the region between the first (singlet) and second (lower band of the doublet) photonic bands (see Fig. 1a of the main text) and can be used to prevent leakage of the photonic modes. Since the meta-crystal remains in a topologically trivial phase as long as $\chi_{xy} = \chi_{yx} = 0$, this gap represents a natural choice for investigation of edge states confined to the interface between topologically nontrivial and trivial systems.

To confine the electromagnetic field to the interface and enable formation of the edge modes we consider a junction between nontrivial photonic meta-crystal with the lattice constant $a_0$ and a trivial photonic meta-crystal ($\chi_{xy} = 0$) with a scaled lattice constant $a_1$ chosen in such a way that band gaps of two systems overlap, as illustrated in Fig. 1SB. As can be seen, for the case of $a_1 = 0.76a_0$, the frequency gap of the topologically nontrivial meta-crystal, which opens due to non-vanishing bi-anisotropy $\chi_{xy} \neq 0$, overlaps with the frequency gap of topologically trivial meta-crystal. This ensures exponential decay of the edge modes into both structures and their confinement to the interface, which is confirmed by numerical simulations presented in Fig. 2 of the main text.

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Fig. 1SB: Spectral matching of the band gaps of the topologically nontrivial (left band diagram) and trivial meta-crystals (right diagram). Frequency gaps due to finite bi-anisotropy $\chi_{xy} \neq 0$ in the nontrivial system overlaps with the frequency gap of the trivial one for $a_1 = 0.76a_0$.

Supplement C: Two examples of specific designs of spin-degenerate bi-anisotropic metamaterials

While many designs are possible, only two specific designs shown in Figs. 1SC and 2SC are considered here. Because both designs achieve high values permittivity and permeability through the resonant response of their constitutive elements, the effective parameters are inevitably dispersive. However, the resonant character and frequency dependence of material’s parameters were shown earlier to have no effect on topological properties of photonic systems with broken time reversal symmetry even though they relied on highly resonant gyromagnetic response [1, 2]. We arrive at the same conclusion for the PTI metacrystals proposed here. The only effect of the dispersion is overall flattening of the band structure and narrowing of the band gap due to the proximity to the resonance.

One additional challenge caused by the material dispersion is that the spin-degeneracy condition ($\epsilon = \mu$) cannot be satisfied for all frequencies. However, we have found (see below) that it is sufficient to match permittivity and permeability only in the vicinity of the Dirac degeneracy. Then, $\epsilon - \mu$ mismatch appears to be sufficiently small over the spectral range of the band gap so that the finite bi-anisotropy still removes the degeneracy and opens a complete band gap. As a result, the metacrystal retains its topologically nontrivial photonic phase with protected edge states.

Design 1. In the first design, we consider the metamaterial formed by a periodic arrangement of split rings and cut wires shown in Fig. 1SC. Both of the resonant elements provide resonant response resulting in high values of permittivity and permeability of the metamaterial. Here we mainly rely on the response of the split rings as they provide the desirable high value of the permittivity, permeability as well as magneto-electric coupling. In general, split rings alone do not provide matched electric and magnetic responses. This problem is resolved by the introduction of the cut wires providing an additional electric polarizability in $x$ and $y$ directions. Combining these two elements within a single unit cell provides a sufficient number of degrees of freedom for the optimization and design of the spin degenerate material.
satisfying condition $\epsilon_\perp = \mu_\perp = 14$ at the desirable frequency. Any significant cross-element interaction can be significantly reduced by their judicious placement with respect to each other as shown in Fig. 1SC. Neglecting interaction among different elements we can assume that effective electric polarizability of the medium is a sum of the responses stemming from cut wires and split rings, respectively. Then the effective in-plane permittivity takes the form:

$$\epsilon_\perp = 1 + \chi_{ee\perp}^{CW} + \chi_{ee\perp}^{SR} = 1 + \frac{N A_{CW}}{\omega_{CW}^2 - \omega^2 + i\Gamma_{CW}\omega} + \frac{N A_{SR}}{\omega_{SR}^2 - \omega^2 + i\Gamma_{SR}\omega}, \quad (1SC)$$

where $N = Q / P^2 P_z$ is the density of the cut wires and split rings, $P$ and $P_z$ are the in-plane and out-of-plane dimension of the unit cell, $Q$ is the number of elements per period along the in-plane direction with dense packing, and resonant frequencies $\omega_{CW}$ and $\omega_{SR}$ can be found from self-capacitance and self-inductance of the resonant elements. The effective magnetic permeability and magneto-electric coupling originate entirely from the response of the split ring and assume the form:

$$\mu_\perp = 1 + \chi_{mm\perp}^{SR} = 1 + \frac{\omega^2 N B_{SR}}{\omega_{SR}^2 - \omega^2 + i\Gamma_{SR}\omega}, \quad \text{and} \quad \chi_{mx} = \frac{N D_{SR} \omega}{\omega_{SR}^2 - \omega^2 + i\Gamma_{SR}\omega}. \quad (2SC)$$

**Fig. 1SC:** (a) Hexagonal lattice of the meta-crystal and its microscopic structure made of mixed split rings and cut-wires providing desirable spin-degenerate and bianisotropic response. (b) Dimensions of the resonant elements. (c) Periodic arrangement and stacking configuration of the metamaterial (not to scale).
Design-specific expressions for the amplitudes \( A_{CW} \), \( A_{SR} \), \( B_{SR} \), and \( D_{SR} \) have been given in the recent literature \([3, 4, 5, 6]\). Note that to achieve in-plane isotropic response of the metamaterial we assume that it consists of stacks of layers with alternating orientation of split rings along \( x \) and \( y \) axes respectively, as shown in Fig. 1SC(c). Due to the flatness of the resonant elements such arrangement also allows to pack them relatively densely by closely spacing them in one of the directions. Thus, the period \( P \) in the direction of dense packing contains \( Q \) layers of the resonant elements. In the other in-plane direction only one particle can be placed per unit cell because of the finite size of the particles. The same is valid for the pitch in the vertical direction \( P_z \).

**Design 2.** An alternative design for a bi-anisotropic spin-degenerate metamaterial employs \( \Omega \)-particles \([7]\) as schematically shown in Fig. 2SC(a). The basic \( \Omega \)-particles (Fig. 2SC(b)) are responsible for both electric, magnetic, as well as bi-anisotropic response of the metamaterial. The advantage of using the \( \Omega \)-medium lies in their ability to match the electric and magnetic responses. A reliable analytical models of \( \Omega \)-particles and \( \Omega \)-media have been developed and verified numerically and experimentally in the microwave frequency range \([8, 9]\). In the approximation of weakly coupled \( \Omega \)-particles (which is assumed hereon) the expressions for the relative in-plane permittivity, permeability, and the magneto-electric coupling assume the form:

\[
\varepsilon_{\perp} = 1 + \frac{N A}{\omega_0^2 - \omega^2 + i\Gamma \omega},
\]

\[
\mu_{\perp} = 1 + \frac{N B \omega_0^2}{\omega_0^2 - \omega^2 + i\Gamma \omega},
\]

\[
\chi_r = \frac{N D \omega}{\omega_0^2 - \omega^2 + i\Gamma \omega},
\]

where \( N = Q/P_2 P_z \) is the density of the \( \Omega \)-particles, \( A = \frac{l^2}{\varepsilon_0 \mu_0} \), \( B = \frac{\pi^2 \mu_0 a^4}{L_0} \), \( D = \frac{2\pi^2 \varepsilon_0 a^4}{c L_0 C_0} \), \( \omega_0 = (L_0 C_0)^{-1/2} \) is the resonant frequency, and \( L_0 = \mu_0 a \left[ \log(8a/r_0) - 2 \right] \) and \( C_0 = \frac{\pi l \varepsilon_0 \varepsilon_M}{\log(2l/r_0)} \) are self-inductance and capacitance of the \( \Omega \)-particle, respectively. The resonance bandwidth is determined by the Ohmic losses in the metallic particles through the expression \( \Gamma = \frac{1}{L_0} \sqrt{\frac{\omega \mu_0}{2\sigma}} a \) where \( \sigma \) is the conductivity. Similar to the case of design 1, stacking of the layers with alternating orientation of \( \Omega \)-particles is used to obtain strong and isotropic in-plane response of the metamaterial, as illustrated in Fig. 2SC(c).
As the next step, the structure has to be optimized by tuning microscopic parameters of the effective medium to achieve desirable matching of the $\varepsilon_\perp$ and $\mu_\perp$ at the value of 14 ($\varepsilon_\perp = \mu_\perp = 14$). It is worth mentioning that a similar matching of permittivity and permeability of bianisotropic medium was previously demonstrated for different application where bianisotropy was undesirable and was eliminated by alternating orientation of every next chiral particle of the medium [10, 11]. In the case of PTI considered here, the bianisotropy plays critical role and has to be preserved. However, the absolute value of the magneto-electric coupling $\chi_r$ can be tuned by changing the ratio of the $\Omega$-particles oriented up and down.

Here we will stick to a particular design of $\Omega$-medium composed of copper particles outlined above. The resulting effective in-plane permittivity and permeability of the optimized structure are plotted in Fig. 3SC. As one can see, with the use of the parameters given in the figure caption, we achieved a
desirable condition of the spin-degeneracy $\epsilon_\perp = \mu_\perp=14$ near the dimensionless frequency 0.425. This is about 4.5% away from the resonant frequency of the $\Omega$-particles.

To confirm that the dispersive, lossy and not perfect match of $\epsilon_\perp$ and $\mu_\perp$ over the broad band does not ruin topological phase, we have applied the first-principle numerical technique to find complex photonic band structure [12] of the PTI. We calculated complex eigenvalues of wavevector for different bands $k^n(\omega) = k^r_n(\omega) + ik^n_i(\omega)$ (here $n$ is the band index) for the metacrystal with the rods made of dispersive and lossy $\Omega$-medium. Regardless of the material dispersion, the double Dirac point with fourfold degeneracy was found in the spectrum at the frequency where $\epsilon_\perp=\mu_\perp$. While TE and TM bands were degenerate at this particular frequency, they gradually diverged away from the Dirac point. As can be seen from Fig.4SC (a), where $k^r_n(\omega)$ is plotted, addition of the bianisotropy resulted in the gap opening and the removal of the Dirac degeneracies. This confirmed that the bianisotropy was sufficient to overcome spectral detuning of the TE and TM modes. The relative bandwidth of the gap induced by the bianisotropy was found to be about 2%.

To verify that the material dispersion and loss do not ruin topologically protected edge states, the complex photonic band structure for a supercell of 50 unit cells, with the magneto-electric coupling reversed at the supercell center, was calculated. Figure 4SC (b) shows band diagram in the vicinity of the former Dirac degeneracy and clearly reveals presence of the two edge modes (red lines) interconnecting two domains of the delocalized modes propagating in the bulk of the metacrystal (shaded regions). We have also calculated the value of the imaginary part of the wavenumber of the edge modes (shown in Fig. 4SC (c)) caused by the presence of the resistive loss in the copper. As $k^\text{edge}_i \sim 0.01a^\text{c}^{-1}$ appears to be significantly smaller than the real part $k^\text{edge}_r \sim 4.2a^\text{c}^{-1}$, we conclude that the edge modes propagate many periods before they decay due to inherent resistive losses in the metamaterial.
Fig. 3SC: In-pane permittivity $\varepsilon_\perp$, permeability $\mu_\perp$, and the magneto-electric coupling $\chi_\perp$ of the $\Omega$-medium with the following geometric parameters: $l = 1$ cm, $a = 1.07$ cm, $2r_0 = 0.7$ mm, $P_{\text{dense}} = 2$ mm, $P_{\text{sparse}} = 3$ cm, $P_z = 5$ cm. Resonant wavelength is $\lambda_0 = 21.7$ cm, and losses $\Gamma = \omega_0/1980$ are calculated with the use of the conductivity of copper $\sigma_{\text{cu}} = 5.96 \times 10^7$ S/m.

Fig. 4SC: (a) Real part of the wavenumber $k_i^\text{R} (\omega)$ for the bulk modes of photonic metacrystal made of dispersive and lossy $\Omega$-medium. (b) Real part of the wavenumber $k_i^\text{edge}$ of the topologically protected edge modes (red lines) at the interface between two PTI metacrystals with the reversed bianisotropy. Solutions corresponding to the delocalized bulk modes are shown by the gray shaded regions. (c) Imaginary part of the wavenumber $k_i^\text{edge}$ of the topologically protected modes.

To conclude, we have found that frequency dependent response, losses in the metamaterial, and imperfect spin-degeneracy due the impossibility to match permittivity $\varepsilon_\perp$ and permeability $\mu_\perp$ over a broad spectral range, do not affect topologically nontrivial character of photonic phase of the PTI.
Effects of interactions among resonant elements

So far all the calculations performed above for the \( \Omega \)-medium assumed that the resonant elements are weakly interacting with each other and the simple analytical expressions given by Eqs. 3SC-5SC are valid. However, as the edge modes have to exhibit non-leaky propagation and good confinement to the interface, high values of the permittivity and permeability are required to ensure that the gap induced by the bianisotropy near the Dirac point is a complete (omnidirectional) band gap. On the other hand, it is also preferable to operate as far away from the resonance as possible to minimize effects of dispersion such as gap narrowing and high loss. These two requirements can be satisfied only in the structures with the resonant elements being densely packed, which inevitably leads to interactions among them. However, effects of such interactions are well predictable and can be taken into account. Interaction of the resonant inclusions constituting the metamaterial gives rise, in first place, to spectral shift of the resonant frequency. This effect alone, however, cannot destroy any of the predicted properties of the PTI but can only result in a spectral shift of the Dirac point and the edge states.

To investigate effect of the interaction on the spectral position of the resonance in \( \Omega \)-medium, we performed fully vectorial eigenvalue study of a periodic array of the \( \Omega \)-particles. Frequency of the resonance corresponding to the lowest-order eigenmode of the \( \Omega \)-particles was tracked as a function of their separation in both directions of the dense (x-direction) and sparse (y-direction) packing. Figure 5SC (a) shows that as the particles get closer in the dense direction, the resonance exhibits redshift. This redshift is a signature of increased inductive coupling among the \( \Omega \)-particles which enhances the net magnetic polarizability of the medium since the magnetic moments of the particles polarize each other stronger as they get closer. In contrast to this, decrease in the separation of the \( \Omega \)-particles along the sparse direction, gave rise to blue shift of the resonant frequency, as shown in Fig.5SC (b). This behavior can be again explained as a result of inductive coupling, which, however, now is destructive since the nearby magnetic moments of the \( \Omega \)-particles depolarize each other giving rise to overall decreased magnetic polarizability of the \( \Omega \)-medium. From our observations we can also conclude that the capacitive coupling between \( \Omega \)-particles plays rather marginal role as it would result into dependencies opposite to those observed in simulations.

As can be seen from Figs. 5SC (a) and (b), despite the significant change in the spacing along both directions, the resonant frequency exhibited a moderate shift not exceeding 12%. Such shift can be compensated in practice by rescaling all the characteristic dimensions of the metamaterial.
Fig. 5SC: Spectral position of the resonant frequency of the bianisotropic \( \Omega \)-medium as a function separation of \( \Omega \)-particles in the direction of (a) dense and (b) sparse packing. Structure parameters are the same as in Fig. 3SC.

Another effect of the interactions stemming from the disparate strength of inductive and capacitive couplings among the particles is the possible unequal change in the effective electric and magnetic polarizabilities of the composite medium, which can result in violation of the condition of spin-degeneracy \( \epsilon_\perp = \mu_\perp \). However, this problem can also be straightforwardly resolved by appropriately scaling parameters \( L \) and \( a \), i.e. dimensions of the \( \Omega \)-particles responsible for their electric and magnetic polarizabilities.

References:


