1. Accidental-degeneracy-induced Dirac point in a photonic crystal with a triangular lattice.

Here, we show that the accidental-degeneracy-induced Dirac point (ADIDP) can also be realized using photonic crystals (PCs) with a triangular lattice. The $\varepsilon$ and $\mu$ of the cylinders are the same as those in the square lattice. The radius is changed to $R' = 0.184a$, but the filling ratio remains almost the same as that in the square lattice case. The dispersions are qualitatively similar to those of the PC with a square lattice shown in the text. Two linear bands and one additional flat band intersect at a triply-degenerate point at the $\Gamma$ point as shown in Fig. S1a and a three-dimensional dispersion plot showing the Dirac cone near the $\Gamma$ point is shown in Fig. S1b.
Fig. S1  a, The band structure of a PC with a triangular lattice, the radius, the relative permittivity and permeability of the cylinder are $R' = 0.184a$, $\varepsilon = 12.5$, $\mu = 1$.  

b, A three dimensional dispersion plot showing frequencies as functions of the wave vectors $k_x$ and $k_y$.

2. Dispersions with and without the accidental degeneracy

Here, we show the band structures of several square lattice photonic crystals (PCs) with different radii of the cylinders (and thus different filling ratios). The band structure with accidental degeneracy and thus ADIDP is shown in Fig. S2a, which has a triply-degenerate point at the $\Gamma$ point at frequency $f = 0.541c/a$. Here, the lattice constant is $a$, the radius, relative permittivity and permeability of the cylinder are set at $R = 0.2a$, $\varepsilon = 12.5$, $\mu = 1$, respectively. Two linear branches (forming a Dirac cone) and a third flat branch intersect at a triply-degenerate point. If the radius of the cylinder deviates from $0.2a$, the triply-degenerate point will break up into a doubly-degenerate point corresponding to two dipolar modes and a single monopolar mode. For example, when $R = 0.19a$, as shown in Fig. S2b, the triply-degenerate point is decoupled into a doubly-degenerate point ($f = 0.566c/a$) and a single mode ($f = 0.54c/a$). While for the case of $R = 0.21a$, the triply-degenerate point is also decoupled, but the doubly-degenerate point ($f = 0.52c/a$) is lower in frequency than the single mode ($f = 0.541c/a$), as shown in Fig. S2c. Here, we note that all the dispersions around the doubly-degenerate point and the single mode at the $\Gamma$ point in Figs. S2b and S2c are quadratic. The Dirac cone emerges as a consequence of accidental degeneracy at a particular radius (Fig. S2a).
Fig. S2 The band structure of the photonic crystals with square lattice for three different radii of the cylinders, \( R = 0.2a \), \( R = 0.19a \), \( R = 0.21a \). The lattice constant is \( a \), the relative permittivity and permeability of the cylinder are \( \varepsilon = 12.5 \), \( \mu = 1 \).

3. Electric field maps for the eigenstates at the Dirac point

In order to understand the physical nature of the eigenstates at the ADIDP, we show the electric field patterns of the eigenstates near the Dirac point with a small \( k \) along the \( \Gamma X \) direction (\( k_x = 0.0476 \times 2\pi/a, k_y = 0 \)). In Figs. S3a and S3b, we show the real and imaginary parts of the electric fields of the lowest frequency state (\( f = 0.527c/a \)) which show that the eigenmode is a combination of a monopole and a transverse magnetic dipole. The field pattern for the highest frequency state (\( f = 0.555c/a \)) (not shown here) is similar, also a combination of a monopole and a transverse magnetic dipole. The real part of the electric field of the flat band is plotted in Fig. S3c and the imaginary part is almost zero. From Fig. S3c, we can see it is a pure dipolar mode with magnetic field parallel to the direction of \( \vec{k} \), i.e. a longitudinal magnetic dipole. The field patterns of the modes near the Dirac point tell us that three branches involve only a mixture of monopolar, dipolar modes. The corresponding magnetic fields \( \vec{H} \) are also shown as vector fields in the figures.
Fig. S3  The color patterns show the $E_z$ and the vector fields show $\vec{H}$ of the eigenstates near the Dirac point with a small $k$ along the $\Gamma X$ direction ($k_x = 0.0476 \times 2\pi/a, k_y = 0$).  

- **a**, The real part of $E_z$ and the imaginary part of $\vec{H}$ at $f = 0.527 c/a$,  
- **b**, The imaginary part of the $E_z$ and the real part of $\vec{H}$ at $f = 0.527 c/a$,  
- **c**, The real part of $E_z$ and the imaginary part of $\vec{H}$ at $f = 0.541 c/a$.

4. Multiple scattering theory (MST) analysis of the ADIDP

Here, we apply multiple scattering theory (MST) to analyze the dispersion bands near the ADIDP at the $\Gamma$ point of the square lattice. Through this analysis, we shall see that the Dirac-cone dispersions accompanied by a flat band can emerge as a consequence of the accidental degeneracy of the dipolar and monopolar modes at the $\Gamma$ point.

A. The MST formulation for the problem

The MST equations can be written as

$$ b_n(i) = D_m(i) \sum_{j \neq i} \sum_n G_{m,n}(i,j) b_n(j), \quad (S1) $$

where $b_n(i)$ and $D_m(i)$ are the Mie scattering coefficient and T-matrix coefficients of angular momentum number $m$ for the $i$th scatterer, respectively. $G_{m,n}(i,j)$ denotes the transformation matrix that transforms the scattered wave of angular momentum number $n$ from the $j$th scatterer, into the incident wave of angular momentum number $n$. 
Applying Bloch’s theorem (i.e. $b_m(i) = b_m \cdot e^{i \vec{k} \cdot \vec{R}_i}$, in which $\vec{k}$ is the Bloch wave vector and $\vec{R}_i$ is the lattice vector of the $i$th scatterer), Eq. (S1) becomes

$$b_m = D_m \cdot \sum_n S_{n-m}(-1)^{n-m} b_n,$$

(S2)

where $S_{n-m}$ is the lattice sum. $S_n$ for $n \geq 0$ can be obtained in the reciprocal space as

$$S_n = \frac{4i^{n+1}k_0}{\Omega} \sum_{G_i} J_{n+1}(k_G r_m) \frac{e^{in\phi_G}}{k_G(k_0^2-k_G^2)} H_1^{(1)}(k_0 r_m) + \frac{2i}{\pi k_0 r_m} \delta_{n,0}. \quad \text{(S3)}$$

Here $G_i$ is the reciprocal lattice vector; $k_0 = \omega/c$; $r_m$ is the nearest neighbor distance; $\Omega$ is the area of the 2D unit cell, and $(k_G, \phi_G)$ are the polar coordinates of the vector $\vec{k} + \vec{G}_i$, i.e. $k_G = |\vec{k} + \vec{G}_i|$ and $\phi_G = \text{Arg}(\vec{k} + \vec{G}_i)$. For $n < 0$, we have $S_n = -S_{-n}$.

From the eigenstate field maps near the ADIDP, we know that they are mainly dipolar ($m = \pm 1$) and monopolar ($m = 0$) modes. Thus, here we can apply the MST to the subspace spanned by dipolar and monopolar modes, which results in a $3 \times 3$ matrix as:

$$\begin{pmatrix}
S_0 - 1/D_{\pm 1} & -S_1 & S_2 \\
-S_1 & S_0 - 1/D_0 & -S_1 \\
-S_2 & -S_1 & S_0 - 1/D_{\pm 1}
\end{pmatrix} \begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = 0. \quad \text{(S4)}$$

In order to obtain the dispersion relations, we do a small $\vec{k}$ ($\vec{k} = \delta \vec{k} = (\delta k, \delta \phi)$) and a small $\omega$ expansion for the eigenmodes near the $\Gamma$ point. In the following, we consider the case for a square lattice. The case of triangular lattice can be derived similarly. First, we do a small $\vec{k}$ expansion as

$$f(\delta \vec{k}) \approx f(0) + \nabla_{\delta \vec{k}} f(0) \cdot \delta \vec{k} + O(\delta \vec{k}^2)$$

for $S_0 - 1/D_{\pm 1}$, $S_{\pm 1}$ and $S_{\pm 2}$ in Eq. S4.
B. Small $\vec{k}$ expansion for $S_0$

In the limit of $\delta \vec{k} \to 0$, $S_0$ can be directly written in the following form:

$$S_0 = -1 + i \left( \frac{4k_0}{\Omega} \left( \frac{r_m}{2J_1(k_0 r_m)} k_0^2 + \sum_{\ell, \omega_0} J_1(k_0 r_m) \left( \frac{1}{k_0^2 - k_{g_i}^2} \right) \right) \frac{Y_1(k_0 r_m) + \frac{2}{\pi k_0 r_m}}{J_1(k_0 r_m)} \right)$$

$$= -1 + i \left( S_0^C + S_0^G \right),$$

where $S_0^C = \frac{4k_0}{\Omega} \left( \frac{r_m}{2J_1(k_0 r_m)} k_0^2 \right) \frac{Y_1(k_0 r_m) + \frac{2}{\pi k_0 r_m}}{J_1(k_0 r_m)}$ is a function of $\omega$ only, and

$$S_0^G = \frac{4k_0}{\Omega} \left( \sum_{\ell, \omega_0} J_1(k_0 r_m) \left( \frac{1}{k_0^2 - k_{g_i}^2} \right) \right)$$

is a function of both $\omega$ and $\delta \vec{k}$. $S_0^G$ is a summation of many terms. Each term is associated with a reciprocal lattice vector $\vec{G}_i \neq 0$. Here, the superscripts $C$ and $G$ indicate the parts associated with reciprocal vectors $\vec{G} = 0$ and $\vec{G} \neq 0$, respectively.

Due to the symmetry of the reciprocal lattice of the square lattice, $S_0^G$ can be greatly simplified. In the square lattice, for any reciprocal lattice vector $\vec{G}_i$ with amplitude $|G_i| = g$ and polar angle $\arg(\vec{G}_i) = \psi_{\vec{G}_i}$, we can always find three other reciprocal lattice vectors $\vec{G}_2$, $\vec{G}_3$ and $\vec{G}_4$, with the same amplitude $g$ and polar angles $\psi_{\vec{G}_2} = \psi_{\vec{G}_i} + \pi/2$, $\psi_{\vec{G}_3} = \psi_{\vec{G}_i} + \pi$, $\psi_{\vec{G}_4} = \psi_{\vec{G}_i} + 3\pi/2$, respectively, as shown in Fig. S4. The reciprocal lattice of the square lattice.
Fig. S4. In the limit of $\delta \bar{k} \to 0$, we have $k_{G_i} = \lvert \delta \bar{k} + \tilde{G}_i \rvert \approx g + \frac{\tilde{G}_i \cdot \delta \bar{k}}{g}$. If we consider $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3$ and $\tilde{G}_4$ in $S_0^G$, then we find

$$f_g = \sum_{i=1,2,3,4} \frac{J_1(k_{G_i}r_m)}{J_1(k_0r_m)} \frac{1}{k_{G_i}(k_0^2 - k_{G_i}^2)},$$
i.e. the summation of the four terms corresponding to $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3$ and $\tilde{G}_4$ in $S_0^G$, then we find

$$f_g = \sum_{i=1,2,3,4} \frac{J_1\left(g + \frac{\tilde{G}_i \cdot \delta \bar{k}}{g}\right)r_m}{J_1(k_0r_m)} \frac{1}{g + \frac{\tilde{G}_i \cdot \delta \bar{k}}{g}} \left(k_0^2 - g^2\right)^{-1/2} \left(1 + \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2} \left(1 - \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2} \left(1 + \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2}.$$

The zeroth-order part of $f_g$ is

$$f_g = \sum_{i=1,2,3,4} \frac{J_1\left(g r_m\right)}{J_1(k_0r_m)} \frac{1}{g \left(k_0^2 - g^2\right)} \frac{4}{\left(1 - \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2} \left(1 + \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2} \left(1 + \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2} \left(1 - \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2} \left(1 + \frac{2\tilde{G}_i \cdot \delta \bar{k}}{k_0^2 - g^2}\right)^{-1/2}}.$$

The first order part is proportional to $\sum_{i=1,2,3,4} \tilde{G}_i \cdot \delta \bar{k}$. However, since we have $\tilde{G}_1 = -\tilde{G}_3$ and $\tilde{G}_2 = -\tilde{G}_4$, we find $\sum_{i=1,2,3,4} \tilde{G}_i \cdot \delta \bar{k} = 0$ and thus the first order part vanishes. Therefore, we obtain $f_g \approx \sum_{i=1,2,3,4} \frac{J_1\left(g r_m\right)}{J_1(k_0r_m)} \frac{4}{g \left(k_0^2 - g^2\right)} + O\left(\delta \bar{k}^2\right)$. For other reciprocal lattice vectors of $\tilde{G}_i \neq 0$, the above analysis also applies. As a result, we obtain $S_0^G \approx f(\omega) + O\left(\delta \bar{k}^2\right)$, where $f(\omega) = \sum_{\tilde{G}_i \neq 0} \frac{J_1\left(\left|\tilde{G}_i\right| r_m\right)}{J_1(k_0r_m)} \frac{1}{\left|\tilde{G}_i\right|^2} \frac{\left(k_0^2 - \left|\tilde{G}_i\right|^2\right)}{g \left(k_0^2 - g^2\right)}$ is only dependent on $\omega$.

Here, we note that $S_0^C$ and $S_0^G$ are both real numbers (Eq. S5). From the optical theorem (see e.g. Ref. S1), we have $\text{Re}(D_n) = -|D_n|^2$ and

$$\text{Im}(D_n) = \frac{\left|D_n\right|^2}{2}.$$
\[
\frac{1}{D_n} = \frac{\text{Re}(D_n) - i \text{Im}(D_n)}{|D_n|^2} = -1 - i \frac{\text{Im}(D_n)}{|D_n|^2}.
\]
The real part of \(1/D_n\) is \(-1\), which is exactly the same as \(S_0\). Therefore, the term \(S_0 - 1/D_n = i \left( S_0^C + S_0^G + \text{Im}(D_n) \right)/|D_n|^2\) is a purely imaginary number. Thus, \(S_0 - 1/D_0\) and \(S_0 - 1/D_{\pm 1}\) as \(\delta k \to 0\) can be approximately written as
\[
S_0 - 1/D_0 = i \left( A_0(\omega) + B(\omega) \delta k^2 \right),
\]
\[
S_0 - 1/D_{\pm 1} = i \left( A_1(\omega) + B(\omega) \delta k^2 \right),
\]
where \(A_0(\omega), A_1(\omega)\) and \(B(\omega)\) are all real functions of \(\omega\) only. Here, we have applied \(D_1 = D_{-1}\), because the cylinders are cylindrically symmetric.

\[\textbf{C. Small } \tilde{k} \text{ expansion for } S_1\]

In the limit of \(\delta \tilde{k} \to 0\), \(S_1\) can be written as
\[
S_1 \approx -4k_0 \frac{\delta k \cdot r_m^2}{\Omega} e^{i\phi_i} + \sum_{\ell, \mu} \frac{J_2(k_0 r_m)}{J_2(k_0 r_m')} \frac{e^{i\phi_i}}{k_{\ell\mu} (k_0^2 - k_{\ell\mu}^2)},
\]
\[
= S_1^C + S_1^G,
\]
where \(S_1^C = -4k_0 \frac{\delta k \cdot r_m^2}{\Omega} e^{i\phi_i}\) and \(S_1^G = \frac{-4k_0}{\Omega} \sum_{\ell, \mu} \frac{J_2(k_0 r_m)}{J_2(k_0 r_m')} \frac{e^{i\phi_i}}{k_{\ell\mu} (k_0^2 - k_{\ell\mu}^2)}\) are both functions of \(\omega\) and \(\delta \tilde{k}\). It is seen that the first part \(S_1^C\) is proportional to \(\delta k \cdot e^{i\phi_i}\). The second part \(S_1^G\) is a summation of many terms like \(S_0^G\).

When \(\delta \tilde{k} \to 0\), we have \(k_{\ell\mu} = |\tilde{k} + \tilde{G}_i| \approx g + \frac{\tilde{G}_i \cdot \delta \tilde{k}}{g}\) and
\[
\phi_i = \text{Arg} \left( \tilde{k} + \tilde{G}_i \right) \approx \psi + \frac{\tilde{G}_i \times \delta \tilde{k} \cdot \hat{z}}{g^2},
\]
where \(\psi = \text{Arg} \left( \tilde{G}_i \right)\). Again, consider
\[
f_g = \sum_{i=1,2,3,4} \frac{J_2(k_0 r_m)}{J_2(k_0 r_m')} \frac{e^{i\phi_i}}{k_{\ell\mu} (k_0^2 - k_{\ell\mu}^2)},
\]
i.e. the summation of the four terms corresponding to \(\tilde{G}_1, \tilde{G}_2, \tilde{G}_3\) and \(\tilde{G}_4\) in \(S_1^G\), then we find...
The zeroth-order part of $f_g$, i.e. $\sum_{i=1,2,3,4} \frac{J_2(g_{r_m})}{J_2(k_{0r_m})} \frac{e^{i\omega \theta_i}}{g(k_0^2 - g^2)}$, is proportional to $\sum_{i=1,2,3,4} e^{i\omega \theta_i}$. Since we have $e^{i\omega \theta_1} = -e^{i\omega \theta_3}$ and $e^{i\omega \theta_2} = -e^{i\omega \theta_4}$, thus the zeroth-order part vanishes. The first order part of $f_g$ involves terms proportional to $\sum_{i=1,2,3,4} e^{i\omega \theta_i} \tilde{G}_i \cdot \delta \vec{k}$ or $\sum_{i=1,2,3,4} i e^{i\omega \theta_i} \tilde{z} \times \tilde{G}_i \cdot \delta \vec{k}$. Because $\sum_{i=1,2,3,4} e^{i\omega \theta_i} \tilde{G}_i \cdot \delta \vec{k} = 2 g \delta k e^{i\Phi}$ and $\sum_{i=1,2,3,4} i e^{i\omega \theta_i} \tilde{z} \times \tilde{G}_i \cdot \delta \vec{k} = 2 g \delta k e^{i\Phi}$, therefore, the first order part $f_g \sim \delta k \cdot e^{i\Phi}$. Similar analysis can be applied for other reciprocal lattice vector terms, thus, we have $S_i^G \sim \delta k \cdot e^{i\Phi}$, as $S_i^C$. Thus, the lattice sum $S_1$ can be written as

$$S(1) = C_1(\omega) \delta k e^{i\Phi},$$

(S8)

where $C_1(\omega)$ is a real function of $\omega$ only.

D. Small $\vec{k}$ expansion for $S_2$

In the limit of $\delta \vec{k} \to 0$, $S_2$ can be written as

$$S_2 = \frac{-4ik_0}{\omega} \left( \frac{\delta k^2 \cdot r_m^3}{48J_3(k_0r_m)k_0} e^{i2\Phi} + \sum_{G_i \neq 0} \frac{J_3(k_{0r_m})}{J_3(k_0r_m)k_{G_i}(k_0^2 - k_{G_i}^2)} e^{i2\delta \theta_i} \right),$$

(S9)
where \( S_2^C = \frac{-4i k_0}{\Omega} \left( \frac{\delta k^2 \cdot r_m}{48 J_3 (k^2) k_0^2} e^{i2\phi} \right) \) and \( S_2^G = \frac{-4i k_0}{\Omega} \left( \sum_{m=0}^{\infty} \frac{J_1(k^2 r_m)}{J_3(k^2 r_m)} \frac{e^{i2\phi}}{k_0^2 (k_0^2 - k^2)} \right) \) are both functions of \( \omega \) and \( \delta \). \( S_2^C \) is proportional to \( \delta k^2 \). \( S_2^G \) is a summation of many terms like \( S_0^G \) and \( S_1^G \).

Again, consider \( f_g = \sum_{i=1,2,3,4} \frac{J_1(k^2 r_m)}{J_3(k^2 r_m)} \frac{e^{i2\phi}}{k_0^2 (k_0^2 - k^2)} \), i.e. the summation of the four terms corresponding to \( \tilde{G}_1, \tilde{G}_2, \tilde{G}_3 \) and \( \tilde{G}_4 \) in \( S_2^G \), then we find

\[
f_g = \sum_{i=1,2,3,4} \left( g + \tilde{G}_i \cdot \delta \tilde{k} \right) \frac{e^{i2\phi}}{g} \left( 1 + 2i \tilde{G}_i \cdot \delta \tilde{k} \cdot \tilde{z} \right) \approx \sum_{i=1,2,3,4} \left( g + \tilde{G}_i \cdot \delta \tilde{k} \right) \frac{e^{i2\phi}}{g} \left( 1 + 2i \tilde{G}_i \cdot \delta \tilde{k} \cdot \tilde{z} \right).
\]

The zeroth-order part of \( f_g \), i.e. \( \sum_{i=1,2,3,4} \frac{J_1(k^2 r_m)}{J_3(k^2 r_m)} \frac{e^{i2\phi}}{g} \), is proportional to \( \sum_{i=1,2,3,4} e^{i2\phi_i} \). Since we have \( e^{i2\phi_i} = -e^{i2\phi_i} = e^{i2\phi_i} = -e^{i2\phi_i} \), the zeroth-order part vanishes. Due to \( \sum_{i=1,2,3,4} e^{i2\phi_i} \tilde{G}_i \cdot \delta \tilde{k} = 0 \) and \( \sum_{i=1,2,3,4} i e^{i2\phi_i} \tilde{z} \times \tilde{G}_i \cdot \delta \tilde{k} = 0 \)

\( ( e^{i2\phi_i} \tilde{G}_1 = -e^{i2\phi_i} \tilde{G}_3 \) and \( e^{i2\phi_i} \tilde{G}_2 = -e^{i2\phi_i} \tilde{G}_4 \), the first order part of \( f_g \) also vanishes. Therefore, only second order or higher order parts exist, which leads to \( f_g \sim O(\delta k^2) \).

Similar analysis applies for other reciprocal lattice vector terms, thus, we have

\( S_2^G \sim O(\delta k^2) \) as \( S_2^C \). Thus, the lattice sum \( S_2 \) can be written as

\[
S(2) = C_2(\omega, \phi_i) \delta k^2,
\]

(S10)

where \( C_2(\omega, \phi_i) \) is a function of \( \omega \) and \( \phi_i \).
E. Doing a small $\omega$ expansion to obtain the dispersion relations

Besides a small $\delta k$ expansion, we also need to do a small $\omega$ expansion to obtain the dispersions of the eigenstates near the $\Gamma$ point. At the $\Gamma$ point, where $\delta k = 0$, we have $S_{x1} = S_{x2} = 0$ due to the symmetry of square lattice. The secular equation of Eq. S4 reduces into three independent equations, i.e. $S_0 - 1/D_0 = 0$, $S_0 - 1/D_1 = 0$, and $S_0 - 1/D_2 = 0$. By solving $S_0 = 1/D_0$, we can obtain the monopolar eigenfrequency $\omega_m$. By solving $S_0 = 1/D_1 = 1/D_2$, we can obtain the dipolar eigenfrequency $\omega_d$. Near $\omega_m$ and $\omega_d$, we can do a small $\delta \omega$ expansion for $S_0 - 1/D_0$ and $S_0 - 1/D_2$ such that the dispersions nearby ($\delta \omega$ as a function of $\delta k$) can be obtained.

a) the general case of $\omega_m \neq \omega_d$

First, we consider the general case of $\omega_m \neq \omega_d$.

Near $\omega_m$, we get $S_0 - 1/D_0 = i[A'(\omega)(\omega - \omega_m) + B(\omega)\delta k^2]$, where $A'(\omega) = \partial A_0(\omega)/\partial \omega$. We also have $S_0 - 1/D_2 = iA_1(\omega)$, in which the $B(\omega)\delta k^2$ term is omitted as it is a high order term. By substituting them into the MST equations in Eq. S4, we obtain

\[
\begin{pmatrix}
    iA_1 & -C_2\delta k^2 & C_1\delta k e^{i\phi} \\
-C_2\delta k^2 & i\left(A'_0(\omega - \omega_m) + B\delta k^2 \right) & -C_1\delta k e^{i\phi} \\
C_1\delta k e^{-i\phi} & -C_2\delta k^2 & iA_1
\end{pmatrix}
\begin{pmatrix}
    b_{i1} \\
    b_0 \\
    b_1
\end{pmatrix}
= 0. \quad (S11)
\]

Here $A'_0, A_1, B, C_1$ and $C_2$ can all be considered as nonzero constants in a small $\omega$ and $k$ regime. By solving the secular equation, we find $A'_0(\omega - \omega_m) + B\delta k^2 = 0$, which means that the dispersion of the monopolar band is quadratic.
Near $\omega_d$, we obtain $S_0 - 1/D_{s1} = i(A'(\omega)\omega - \omega_d) + B(\omega)\delta k^2$, where

$$A'(\omega) = \partial A(\omega)/\partial \omega.$$ We also have $S_0 - 1/D_0 = iA_0(\omega)$, in which the $B(\omega)\delta k^2$ term is omitted as it is a high order term. By substituting them into the MST equations in Eq. S4, we obtain

$$\begin{bmatrix}
i(A'(\omega - \omega_d) + B\delta k^2) & -C_0\delta k^2 & C_2\delta k^2 \\
C_0\delta k^2 & iA_0 & -C_0\delta k^2 \\
-C_0\delta k^2 & C_0\delta k^2 & i(A'(\omega - \omega_d) + B\delta k^2)
\end{bmatrix} \begin{bmatrix}
b_1 \\
b_0 \\
b_1
\end{bmatrix} = 0. \quad (S12)$$

Here $A_0$, $A'$, $B$, $C_1$ and $C_2$ can be taken as nonzero constants in a small frequency region. By solving the secular equation, we find $A'(\omega - \omega_d) + B\delta k^2 = \pm |C_2|\delta k^2$, which means that the dispersions of the dipolar bands are also quadratic.

**b) the accidental degeneracy case of $\omega_m = \omega_d = \omega^*$**

Now, we consider the special case of $\omega_m = \omega_d = \omega^*$ due to accidental degeneracy.

Near $\omega^*$, we obtain $S_0 - 1/D_0 = i(A'(\omega)\omega - \omega^*) + B(\omega)\delta k^2$, where $A'_0(\omega) = \partial A_0(\omega)/\partial \omega$, and $S_0 - 1/D_{s1} = i(A'(\omega - \omega_d) + B(\omega)\delta k^2)$, where $A'(\omega) = \partial A(\omega)/\partial \omega$. By substituting them into the MST equations in Eq. S4, we obtain

$$\begin{bmatrix}
i(A'(\omega - \omega^*) + B\delta k^2) & -C_0\delta k^2 & C_2\delta k^2 \\
C_0\delta k^2 & iA_0 & -C_0\delta k^2 \\
-C_0\delta k^2 & C_0\delta k^2 & i(A'(\omega - \omega^*) + B\delta k^2)
\end{bmatrix} \begin{bmatrix}
b_1 \\
b_0 \\
b_1
\end{bmatrix} = 0. \quad (S13)$$

Here $A'_0$, $A'$, $B$, $C_1$ and $C_2$ can be considered as nonzero constants in a small frequency regime.

The secular equation of Eq. S13 is a cubic equation of $\delta \omega$:

$$-\left(A'_0\delta \omega + B\delta k^2\right)^2\left(A'_0\delta \omega + B\delta k^2\right) + 2\left(A'_0\delta \omega + B\delta k^2\right)C_2^2\delta k^2
+\left(A'_0\delta \omega + B\delta k^2\right)\left|C_2^2\right|\delta k^4 - 2\text{Im}\left(C_2^2\delta k^2\right)C_1^2\delta k^4 = 0, \quad (S14)$$
where $\delta\omega = \omega - \omega^*$. By solving this secular equation, we find three solutions: one is $\omega_1 - \omega^* = 0 + O(\delta k^2)$, which corresponds to a band of quadratic dispersion (being flat near the $\Gamma$ point). The other two are $\omega_{2,3} - \omega^* = \pm v_g \delta k + O(\delta k^2)$, where $v_g = \frac{\sqrt{2}|C_1|}{\sqrt{A'A_0}}$, which correspond to two bands with Dirac-like linear dispersions.

By substituting $\omega_1 - \omega^* = 0$ into Eq. S13, and omitting the high order terms of $O(\delta k^2)$, we find $b_0 = 0$ and $b_1 e^{-i\phi_0} - b_1 e^{i\phi_0} = 0$. The scattered field can be written as $E_{sc} = b_1 H_1(k_0 r) e^{-i\phi + b_1 H_1(k_0 r) e^{i\phi}} = 2ib_1 e^{i\phi} H_1(k_0 r) \sin(\phi - \phi_k)$, indicating a magnetic dipole oriented parallel to $\delta \vec{k}$. Such a dipole can be viewed as a longitudinal magnetic plasma mode, as it has a magnetic field parallel to $\delta \vec{k}$ and is a result of effective zero permeability.

By substituting $\omega_{2,3} - \omega^* = \pm v_g \delta k$ into Eq. S13 and omitting the high order terms of $O(\delta k^2)$, we find $\pm i v_g A' b_{-1} - C_1 e^{i\phi} b_0 = 0$ and $\pm i v_g A' b_1 + C_1 e^{-i\phi} b_0 = 0$. This indicates that the dipolar and the monopolar modes are coupled together in the Dirac cone.

5. The longitudinal band

The longitudinal band is taken into consideration in our effective medium theory and the physical origin of this flat band is a magnetic longitudinal band induced by $\mu_\ell = 0$. This flat band must exist if we have a $\mu = 0$ system and the existence of such a band in the band structure of the photonic crystal at the Dirac Cone is consistent with the fact that the photonic crystal has an effectively zero index. The band is dispersionless if the $\mu = 0$ system is perfectly homogeneous and as a longitudinal mode, it is a “deaf band” which does not couple with external light. But in any real photonic crystal or metamaterial with $\mu_\ell = 0$, which comprises discrete
building blocks, there is always some spatial dispersion, so that the band is not perfectly dispersionless away from the zone center and the flat band can be excited if light is incident with non-zero k-parallel components.

In our simulations and experiments, we applied an incident plane wave at normal incidence, which cannot excite the flat band. At the Dirac point frequency, the flat band can be excited if we use a tightly focused Gaussian beam incidence which carries non-zero k-components along the surface. The transmitted light at the exit surface is not plane wave like and the wave pattern is due to the flat band excited inside the photonic crystal, as shown in Fig. S5a.

But since this magnetic longitudinal mode has a narrow band width, we can avoid this band (if we want to) by operating at slightly above the Dirac frequency, where the $n_{\text{eff}}$ is still very close to zero. Without the excitation of the longitudinal band, the output wave has a plane-wave wavefront again, as shown in Fig. S5b, the same as a hypothetical homogeneous material with near zero refractive index.

We note again that this flat band must be there if the material has effective permittivity and permeability both equal to zero.

Fig. S5: The simulated $E_z$ field distribution with a tightly focused Gaussian beam incidence, focused at the left interface of our PC, at the Dirac frequency ($f=0.541c/a$)
a and slightly above the Dirac frequency \((f=0.56c/a)\). Here, “a” is the lattice constant of the PC, \(c\) is the velocity of light in vacuum. The parameters of the structure are the same as Fig. 1 in the text.

6. The cloaking effect for PEC and dielectric objects in the photonic crystal system with ADIDP

In the text, we demonstrated that wave can pass through the photonic crystal system at the ADIDP with an embedded PMC object inside it (Fig. 3b). It is treated as a cloaking effect in the literature [S2]. For completeness, we numerically illustrate similar effect for embedded PEC and dielectric objects. As a control calculation, we replace the PC with ADIDP by the homogeneous high dielectric medium \((\varepsilon = 12.5)\), the incident waves are almost totally reflected back by the 90 degree bending channel without any guiding effects, as shown in Fig. S6a. We now put a PEC object or dielectric object \((\varepsilon = 6)\) in a 90 degree bending channel filled with the PC system with ADIDP, the electromagnetic waves can still transmit through with little distortions, as are shown in Figs. S6b and S6c, respectively. This is due to the effective zero permittivity and zero permeability [Ref. S3]. Similar effect can also be observed in single zero material [Ref. S4].
Fig. S6 Simulated electric field ($E_z$) patterns in the 90 degree bending channel. \(a\), $E_z$ distribution if the 90 degree bending channel is filled with a homogeneous high dielectric medium ($\varepsilon = 12.5$). Wave incident from the lower left hand channel is reflected. $E_z$ distribution if the 90 degree bending channel is filled with PCs with an embedded PEC object \(b\) and a dielectric object ($\varepsilon = 6$) \(c\). The incident wave is plane wave and the boundary conditions of the channel are PMC. The working frequency is $0.541c/a$.

References

