Supplementary material for: Boundary layer control of rotating convection systems

Eric M. King\textsuperscript{1,}\textsuperscript{*}, Stephan Stellmach\textsuperscript{2,3}, Jerome Noir\textsuperscript{1}, Ulrich Hansen\textsuperscript{2} and Jonathan M. Aurnou\textsuperscript{1}

\textsuperscript{1}Department of Earth and Space Sciences, University of California, Los Angeles, 90095-1567 USA

\textsuperscript{*}Corresponding Author, email: eric.king@ucla.edu

\textsuperscript{2}Institut für Geophysik, WWU Münster, AG Geodynamik Corrensstr. 24, Münster, 48149 Germany

\textsuperscript{3}Now at: Department of Applied Mathematics and Statistics, and Institute of Geophysics and Planetary Physics, University of California, Santa Cruz, 95064 USA
1 Methods

Laboratory experiments are carried out in water ($Pr \approx 7$) and sucrose solution ($Pr \approx 10$) in a 20 cm diameter right cylinder with diameter to height ratio, $\Gamma$, varying from 6.25 to 1. The tank is heated from below by a non-inductively wound electrical heating coil and rotated at up to 50 rotations per minute about a vertical axis. The temperature difference across the fluid layer, $\Delta T$, is measured by two six-thermistor arrays imbedded in the top and bottom boundaries. The heat flux, $Q$, is calculated by comparing the power input to the resistor with the heating rate of coolant cycling through a thermostated bath atop the convection tank. The Biot number, $Bi$, characterizes the effective boundary to fluid conductance ratio:

$$Bi = Nu \frac{k_{\text{fluid}} H_{\text{boundary}}}{k_{\text{fluid}} H_{\text{boundary}}}$$

where $k$ is thermal conductivity and $H$ is layer thickness. For our experiments, $Bi < 0.1$, and therefore the effects of the finite conductivity in the boundaries are small. The experiment is surrounded by more than 10 cm of closed-cell foam insulation to minimize sidewall heat loss.

Numerical experiments. Direct numerical simulations are carried out by solving the Boussinesq Navier-Stokes equation, energy equation, and continuity equation in a cartesian box for $Pr = 1$, $Pr = 7$, and $Pr = 100$. The box has periodic sidewalls; impenetrable, no-slip top and bottom boundaries; resolution up to $384 \times 384 \times 257$; and aspect ratio $\Gamma$ varying from 1 to 4. Gravity and the rotation axis are both vertical. Fourier transform methods are employed in the horizontal directions, and Chebyshev polynomials are used in the vertical direction.

2 The Boundary Layer

Supplementary figure 1 shows vertical profiles of mean temperature, temperature variance, and velocity. Isothermalization of the interior fluid for $Ra/Ra_t > 1$ permits the formation of
Supplementary figure 1: Vertical profiles of a) mean temperature (non-dimensionalized by the total temperature drop, $\Delta T$), b) temperature variance, and c) RMS velocity (non-dimensionalized by $\kappa/D$, and normalized by $Ra^{1/2}$ for comparison) for $E = 10^{-4}$, $Pr = 7$, and $1.9 \times 10^6 \leq Ra \leq 2.1 \times 10^8$ from numerical experiments. When $Ra/Ra_t > 1$, thermal boundary layers are well-defined and the interior fluid is nearly isothermal. The thermal boundary layer thickness is well described by the height of the peak value of the temperature variance$^{3-6}$. The height of the peak values of the RMS velocity correspond to the Ekman layer thickness$^{3-6}$.

a well defined thermal boundary layer. The thickness of this boundary layer is well described by the height of the peak value of the temperature variance$^{3-6}$, physically corresponding to the location of the development and departure of thermal plumes before they are mixed in the turbulent interior$^3$.

The Nusselt number, $Nu$, is defined as the ratio of total heat flux to overall conductive heat flux. The overall conductive heat flux is $Q_{\text{conductive}} = k \Delta T/D$, where $k$ is the fluid’s thermal conductivity, $\Delta T$ is the total temperature drop, and $D$ is the height of the layer. In the quasi-static boundary layer, heat transfer is entirely conductive, and an isothermal interior means half the total temperature drop occurs in each (top and bottom) boundary layer. Since the total heat flux must be the same through all horizontal planes of infinite extent, the heat flux through the boundary layer equals the total heat flux,
\( Q_{\text{Total}} = k \Delta T / 2 \delta_\kappa \), where \( \delta_\kappa \) is the thermal boundary layer thickness. The Nusselt number is then \( Nu \equiv Q_{\text{Total}} / Q_{\text{conductive}} \approx D / 2 \delta_\kappa \). Supplementary figure 2 shows both \( Nu \) and \( D / 2 \delta_\kappa \) from the \( E = 10^{-4} \), \( Pr = 7 \) numerical experiment. The agreement between \( Nu \) and \( D / 2 \delta_\kappa \) illustrates that the heat transfer is, in fact, well-approximated by this thermal boundary layer scaling.

Supplementary figure 2: Nusselt number versus Rayleigh number for \( E = 10^{-4} \), \( Pr = 7 \) numerical experiments. Also shown are thermal boundary layer data \( D / 2 \delta_\kappa \), where \( \delta_\kappa \) is the thermal boundary layer thickness, defined as the height of the peak value of the temperature variance. The agreement between heat transfer measurements and boundary layer estimates shows that the boundary layer scaling of the Nusselt number, \( Nu \approx D / 2 \delta_\kappa \), is a valid approximation.

3 Transition Scaling

The Rossby number, \( Ro \), characterizes the strength of inertia versus Coriolis acceleration, \( Ro = U / 2 \Omega D \), where \( U \) is a typical fluid velocity, \( \Omega \) is the angular rotation rate, and \( D \) is the length scale of the system. When \( Ro \ll 1 \), it is thought that rotation will domi-
nate convection dynamics$^{7-11}$. The Taylor-Proudman constraint dictates that interior flow structures will be roughly axially-invariant, and therefore quasi-two dimensional in this dynamical regime$^{11}$. Viscosity will only become important at small ($O(E^{-1/3})$) length scales in the horizontal direction$^{12}$, and so we anticipate long, thin, axial flow structures when $Ro \ll 1$. Unfortunately, $U$ is not directly observable in remote convection systems such as planetary interiors, and is not known a priori in experiments and simulations. The convective Rossby number, $Ro_c$, circumvents this difficulty by assuming inertia scales with buoyancy, thus employing the free-fall velocity assumption, $U_{\text{free-fall}} \approx \sqrt{\alpha_T g \Delta T D}$, where $\alpha_T$ is the fluid’s thermal expansion coefficient and $g$ is gravitational acceleration. By using $U_{\text{free-fall}}$, the Rossby number becomes the convective Rossby number, $Ro_c = \sqrt{\alpha_T g \Delta T / 4 \Omega^2 D}$, which is the gauge typically used to predict the importance of rotation in convection systems$^{8-10}$.

According to this global force balance argument, strongly three-dimensional flow structures should not be manifested by convection with $Ro_c < 1$. We have observed, however, three-dimensional convection for $Ro_c < 1$ in our numerical experiments (Fig. 1, main text). Furthermore, heat transfer is well described by the boundary layer controlled $Ra_t$, and not by $Ro_c$ (Fig. 4, main text). This indicates that $Ro_c$ does not adequately describe the influence of rotation in convection systems.

Several other heat transfer scalings for rotating convection have been put forth. Ref. 13 develops a turbulent scaling model which predicts a transitional Rayleigh number that scales as $E^{-6/4}$. Currently, it is difficult to determine which scaling, $Ra_t \sim E^{-7/4}$ or $E^{-6/4}$, better describes the transition. In order to resolve this point, experiments must attain higher $Ra$ with lower $E$. Nevertheless, in the development of the $E^{-6/4}$ transition scaling, Ref. 13 utilizes a $Nu \sim Ra^{1/3}$ non-rotating law, which is not supported by experimental studies.

Ref. 14 proposes the application of weakly non-linear theory to convective heat transfer for $Ra/Ra_t < 1$, which predicts $Nu \sim Ra/Ra_c$. However, we find the steeper, empirical
$\text{Nu} = (\text{Ra}/\text{Ra}_c)^{6/5}$ scaling better fits our data, as in many rotating convection and dynamo simulations in spherical shells\textsuperscript{15,16}.

Supplementary figure 3: Boundary layer control of heat transfer transitions for $10^{-5} \leq E \leq 10^{-3}$ and $1 \leq Pr \leq 100$ from numerical experiments. Circles depict $\text{Nu}$ normalized by the non-rotating scaling law, $\text{Nu} = 0.16 \text{Ra}^{2/7}$. Stars represent the ratio of the thermal boundary layer thickness to the Ekman boundary layer thickness. The Ekman layer thickness is defined as the mean height of the peak value of the RMS velocity$^{3-5}$. Three different $Pr$ values are shown: $Pr = 1$, dotted lines; $Pr = 7$, dashed lines; $Pr = 100$, solid lines. The heat transfer data cross unity approximately where the boundary layer ratios cross unity. This supports our hypothesis, which predicts that heat transfer will conform to the non-rotating behavior when the thermal boundary layer becomes thinner than the Ekman boundary layer. Note also that the first order behavior of these quantities is independent of $Pr$, in further agreement with our boundary layer transition hypothesis.
In the main text fig. 3b, we show that the boundary layers cross when $Ra \approx Ra_t$, as predicted, for the $E = 10^{-4}$, $Pr = 7$ numerical case. To show more generally the relationship between the boundary layers and heat transfer, we must look at several different $E$ and $Pr$ values. In supplementary figure 3, we plot $Nu$, normalized by the non-rotating scaling, versus $Ra$ for $10^{-5} \leq E \leq 10^{-3}$ and $1 \leq Pr \leq 100$ from numerical experiments. The heat transfer transition occurs where this ratio, $Nu/N_{u_{Non-Rotating}}$, crosses unity. In the same figure, we plot the the ratio of the thermal boundary layer thickness to the Ekman boundary layer thickness. For each Ekman number, the heat transfer transition occurs where the boundary layers cross. Also notable is that the Prandtl number has no first-order effect on this behavior. These results support our hypothesis that boundary layer dynamics control heat transfer transitions across a broad range of $E$ and $Pr$.

### 4 Free-Slip Boundary Simulations

Ref. 17 shows that the thermal wind balance, the phenomenon responsible for ‘spinning up’ plumes to form Taylor columns in systems with free-slip boundaries, becomes important at a distance from the boundary that scales as $E^{1/2}$. This free-slip version of the Ekman layer, called the thermal Ekman layer, is the vertical distance from the boundary over which rotation is able to respond to density perturbations and influence plume formation. Ref. 7 shows further that the boundary layer response to lateral variations of temperature is identical for no-slip and free-slip boundary conditions. Since lateral temperature variations are necessary for thermal convection to occur, the thermal Ekman layer is likely to be dynamically important in rotating convection systems with free-slip boundaries. We further hypothesize, then, that the boundary layer control arguments put forth in the main text should apply to rotating convection systems with free-slip boundaries.
Supplementary figure 4: Heat transfer in rotating convection with free-slip boundaries, $10^{-6} \leq E \leq \infty$, and $1 \leq Pr \leq 100$. a) The Nusselt number as a function of the Rayleigh number. Non-rotating heat transfer follows a $Nu = 0.33Ra^{2/7}$ scaling. Rotating convection follows $Nu = (Ra/Ra_c)^{6/5}$ for $Ra/Ra_t < 1$. b) The Nusselt number normalized by the non-rotating scaling is shown versus $Ra/Ra_t$, where $Ra_t = 5.3E^{-7/4}$. When $Ra/Ra_t > 1$, rotating heat transfer conforms to the non-rotating scaling.

We have carried out a complementary suite of numerical rotating convection experiments with free-slip boundaries. Heat transfer data from the free-slip simulations are shown in supplementary figure 4a. Non-rotating heat transfer is well described by $Nu = 0.33Ra^{2/7}$. Rapidly-rotating convective heat transfer, as in the no-slip experiments, is adequately described by $Nu = (Ra/Ra_c)^{6/5}$, where $Ra_c$ is the critical Rayleigh number for the onset of convection. We use $Ra_c = 9E^{-4/3}$ for free-slip boundaries\(^1\). For sufficiently large $Ra$, rotating convective heat transfer switches to the non-rotating scaling. We define the transitional Nusselt and Rayleigh numbers, $Nu_t$ and $Ra_t$, as the point of intersection between the two scalings, yielding $Nu_t = 0.53E^{-1/2}$ and $Ra_t = 5.3E^{-7/4}$. Supplementary figure 4b shows that, as for the no-slip experiments, the transition in heat transfer behavior is well described by the transitional Rayleigh number, $Ra_t$.

This transition scaling agrees with the boundary layer controlled scaling derived in the
main text. That the boundary layer controlled transition pertains to free-slip convection suggests that our results can be applied to convection systems that are not contained by rigid boundaries, such as stars and gas planets.

References


