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A meeting of mathematical minds

Forty years after its discovery, the quantum Hall effect continues to reveal its secrets to mathematicians and physicists.

At a lecture in 1939, Paul Dirac said that "pure mathematics and physics are becoming ever more closely connected". He went on to say that the two subjects might unify, with "every branch of pure mathematics then having its physical application".

Dirac's prognosis was, and remains, highly speculative. Today, there is no question of a unification of these fields. Techniques from pure mathematics are used in economics, engineering and finance, but there's no sense in – nor reason for – these fields becoming one.

Dirac's sentiment rankles with pure mathematicians because it suggests that physicists regard mathematics more as a tool with which to study the natural world than as a discipline in its own right. Such a view can be a barrier to fruitful collaboration. But when mathematicians and physicists do attempt to solve problems on equal terms, the results can be sublime – as we have seen in the physics of materials and in topology, a branch of pure mathematics that studies shapes and how they are arranged in space.

Mathematicians and physicists working in these fields have made lasting contributions to understanding the quantum Hall effect, which was discovered during a transformative experiment 40 years ago[1,2]. How they achieved this holds lessons for the way in which disciplines – and not only those in the physical sciences – could more successfully engage with each other on common problems.

> **"Perhaps the greatest incentive physicists can create is to leave a problem partially solved."**

Quantum jumps

The quantum Hall effect describes the process through which electrical resistance can be precisely measured in layers of material a few atoms thick.

The original Hall effect, discovered in 1879 by physicist Edwin Hall, describes how a magnetic field applied perpendicularly to a metal strip causes electrons to gather along both ends of the strip, creating a voltage[3]. A century later, the physicist Klaus von Klitzing went further[1]. Working at low temperatures with atomically thin layers of crystalline materials – known as two-dimensional electron systems – he discovered that this voltage is quantized. That is, the voltage changes in jumps, as the applied magnetic field changes. This phenomenon is the quantum Hall effect.

The ability to precisely measure resistance comes from von Klitzing's discovery that resistance is quantized at values that are proportional to a combination of two fundamental physical constants: the charge of the electron and Planck's constant. Moreover, the value of the quantized resistance is accurate even when materials contain impurities, which would otherwise change the resistance. Because of this, the quantum Hall effect is used to confirm the accuracy of the ohm, the unit of electrical resistance. Von Klitzing received the Nobel Prize in Physics for this discovery in 1985, five years after his paper was published.

But what of pure mathematics, and how did topology become involved? It turns out that, at the time, physics was unable to fully explain why the resistance changes in discrete steps when the magnetic field changes. Two years after von Klitzing's discovery, physicist David Thouless provided an explanation using topology[4]. His work was subsequently built on by others, and, in 2016, he was awarded a share of the Nobel Prize in Physics.

But some mathematicians were not satisfied with the standard of proof offered by the physicists, and quantum Hall resistance was added to a famous list of unsolved problems in mathematical physics.

It was not until 2015 – 33 years after Thouless's calculation – that a more-rigorous mathematical proof was published[5] by the mathematician Spyridon Michalakis, at the California Institute of Technology in Pasadena, and physicist Matthew Hastings at Microsoft Research in Santa Barbara, California. The pair began working on the problem in 2008, as Michalakis wrote in *Nature Reviews Physics* earlier this month[6].

Theoretical physicists and mathematicians knew that the average curvature of a geometric object – such as its surface – has a topological nature. They also knew that small local deformations affect the curvature locally. But a more rigorous explanation of quantized Hall resistance needed the theory to extend to global curvature. This is what Michalakis and Hastings achieved, making the link between topology and the quantum Hall effect ironclad.

And the story isn't over, by any means. Topology has been getting more attention from physicists, and from funders such as the Simons Foundation in New York City, which is supporting mathematicians and physicists working on difficult problems, such as the fractional quantum Hall effect. In this phenomenon, complex electron interactions cause the Hall resistance to be quantized at a value that is just a fraction of the charge of the electron.

Rather than seek to unify the two disciplines, as Dirac proposed, perhaps the greatest incentive that physicists can create for mathematicians is to leave a problem partially solved. Ultimately, the mathematical proof for quantum Hall resistance might well not have come about had the question not been classified as one of mathematical physics' unsolved problems.

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