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SUPPLEMENTARY INFORMATION
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Spontaneous recovery in dynamical networks

A) Model: Additional Technical Details and Discussion

Here we provide a more extensive discussion of the technical details of the model. The model is based on the three fundamental assumptions: (i) that nodes in a network can fail due internal causes, (ii) that they can also fail due to external causes, and (iii) that nodes have a recovery process.

(i) We assume that any node can fail randomly and independently of other nodes due to internal causes. We model internal failure using a parameter p. Each node has a probability of internal failure $p\,dt$ during a time interval dt. Internal reasons for failure can be related to any process essential to node integrity.

(ii) We assume that any node can fail due to external causes. For example, if the neighborhood surrounding node j (i.e., a collection of nodes directly connected to j) is substantially damaged, it can negatively impact node j. We use a simple threshold rule (similar to that proposed by Watts [1]) to define a *substantially damaged* neighborhood, i.e., a neighborhood containing fewer than or equal to m active nodes, where m is a fixed integer. For externally-induced failures we assume the following: If node j has more than m active neighbors during the time interval dt, its neighborhood is "healthy" and node j is not at risk of externally-induced failure, but if node j has fewer than or equal to m active neighbors during the interval dt, there is a probability $r\,dt$ that node j will experience externally-induced failure. A "damage conductivity" parameter r describes how rapidly damage spreads through the network.

(iii) We assume that there is a process of recovery after node failure. We use a parameter $\tau \neq 0$ to model the recovery from internal failures. Node j recovers from an *internal* failure after a time period τ. If a node has already failed internally and, before it has recovered, a new internal failure hits this node at t, then the opportunity for recovery will occur at $t+\tau$. A node recovers from an *external* failure after time τ', which usually is not equal to τ. For simplicity, we use $\tau' = 1$ in our simulations. In addition to using

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recovery times, we can also use recovery *rates* to describe the recovery process, similar to that in engineering networks with a "wear-out" process [2] or in forest fire models [3, 4] where vegetation can regrow after it has been burned.

B) Average fraction of internally failed nodes

Let $0 < u(t) < 1$ be the fraction of nodes that are in the internally failed state, where t is a discrete time step. Imagine that on each internally failed node a small clock is activated to measure the time ℓ passed since the last internal failure of the node. At any moment t, each node has a probability $p \ll 1$ to experience a new internal failure, and the clock in the node is then reset to 0. When ℓ reaches the value $\ell = \tau$, the node recovers from the internal failure. Let $c_\ell(t)$ denote the fraction of nodes in which the most recent internal failure occurs at moment $t - \ell$. The evolution of $c_\ell(t)$ and $u(t)$ is given by equations

$$c_\ell(t+1) = (1-p)c_{\ell-1}(t), \tag{1}$$

and

$$u(t+1) - u(t) = p(1-u(t)) - c_\tau(t). \tag{2}$$

The right side of Eq. (1) accounts for the process of aging of time ℓ (nodes in $c_{\ell-1}$ transfer to c_ℓ after one time step) and the factor $(1-p)$ accounts for internal failures during the time step. The boundary condition for $\ell = 0$ is $c_0(t) = p$. In Eq. (2), the first term on the right hand side describes the failure rate of internally-active nodes that contribute to the growth of u, while the second term accounts for the decrease in u due to node recovery when $\ell = \tau$. When the system reaches a steady state, $c_\ell = (1-p)c_{\ell-1}$, with solution $c_\ell = c_0(1-p)^\ell \approx pe^{-pl}$, while the steady state equation for u, $p(1-u) - c_\tau = 0$, gives the solution $u = 1 - e^{-p\tau} \equiv p^*$. This result is for the average fraction of internally failed nodes in the network.

C) Connections to other models

To characterize the system in terms of its phase diagram, we measure the standard critical exponents β, δ, and γ around the critical point (for definitions see Ref. [5]). Large random networks can be treated as infinite dimensional systems. For example, for our prototypical regular ($k = 10$, $m = 4$) network with $N = 10^7$ nodes we find $\beta = 0.5 \pm 0.1$, $\delta = 2.7 \pm 0.6$, and $\gamma = 1.2 \pm 0.3$, indicating that our model is in the mean field Ising universality class in $d \geq 4$ dimensions, i.e., in systems in which $\beta = 1/2$, $\delta = 3$, and $\gamma = 1$. In contrast to magnetic systems and fluids, the critical exponent α, related to the heat capacity, is not defined for our dynamic network because there is no proper equivalence for heat or energy in our system. Note that the pattern of internal failures in our model is analogous to the external magnetic field in lattice-magnetic models (e.g., the Ising model) or the chemical potential in lattice-gas models. The *external* failures correspond to the interaction between neighboring sites in lattice-gas models or neighboring spins in lattice-magnetic models. The phase diagram of our system is rotated when compared to the phase diagram of the Ising model and is similar to the phase diagram of lattice gas models.

Our network model also resembles forest-fire models. In a lattice forest-fire model [3], trees grow with probability p from empty sites at each time step, and the fire on a site (node) will spread deterministically to trees at its neighbor sites at the next time step, with probability $r = 1$ in our terms. Allowing a certain fraction of trees to be immune [4] produces a continuous phase transition from a steady state with fire to a steady state without fire.

D) Estimates for lifetimes $T_{\text{down}}(N)$ and $T_{\text{up}}(N)$

Here we derive formula (2) from the manuscript and the corresponding formula for $T_{\text{up}}(N)$, the estimates for the average lifetimes $T_{\text{down}}(N)$ and $T_{\text{up}}(N)$ of the system in each of the two states. The dependence of $T_{\text{down}}(N)$ and $T_{\text{up}}(N)$ on the system size N can be found using a simple model. First, note that the "left" spinodal is almost vertical (its direction is mostly along the p^*-axis), and the "right" spinodal is almost horizontal (mostly along the r-axis). This geometrical property allows us to approximate the problem with a combination of two independent one-dimensional processes. For $T_{\text{down}}(N)$, the system is in the low active state, and we assume the transition to the high active state occurs when r_λ reaches some value $r_s < r$, where r_s is a typical r-position on the left spinodal where most cascades to the high active state occur. This is essentially a first passage process. Let $E[a(r, p^*)]$ be the average fraction of CDN nodes. This quantity can be calculated summing up $E_k = \sum_{j=0}^{m} \binom{k}{k-j} a^{k-j}(1-a)^j$ for all k (see the Methods section in the manuscript), weighted by corresponding degree distribution factors. There are $NE(a(r, p^*), k, m)$ nodes in the network located in critically damaged neighborhoods, and each has a probability r of failing externally and a probability $1 - r$ of not failing externally. Since we define r_λ as

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the average fraction of externally failed nodes among the nodes located in critically damaged neighborhoods during an interval of length λ, the probability distribution of r_λ values is binomial and can be approximated with the normal distribution with mean $\mu = r$ and variance $\sigma^2 = r(1-r)/n$,

$$f(r_\lambda) \sim \exp[-\frac{n(r_\lambda - r)^2}{2r(1-r)}], \tag{3}$$

where n is the sample size. Since $n = NE(a(r,p^*),k,m)\lambda$, the probability that $r_\lambda = r_s$ is

$$f(r_s) \sim \exp[-\frac{N\lambda E(a(r,p^*),k,m)(r_s - r)^2}{2r(1-r)}]. \tag{4}$$

If there is a probability $f(r_s)$ that at random moment t the trajectory $(r_\lambda(t),\ p_\lambda{}^*(t))$ crosses the "left" spinodal, then the estimate for $T_{\rm down}(N)$ is $T_{\rm down}(N) \sim 1/f_{r_s}$, and we arrive at Eq. (2) from the manuscript.

The equation for $T_{\rm up}(N)$ can be obtained in an analogous manner. These are very simple estimates but they give correct parameter dependencies.

Note that the fluctuations in r become very large when the system is in the upper state. This is the case because the value of the sample size $n = NE(a(r,p^*),k,m)\lambda$ is small, since $E(a(r,p^*),k,m)$ is small in the upper state. Here the trajectory of the system $(r_\lambda(t),\ p_\lambda{}^*(t))$ frequently crosses the "left" spinodal and *nothing special* occurs when it does, since the *"right"* spinodal is relevant for the transition when the system is in the upper state.

E) "Flash crashes" in real-world networks

In addition to phase switching, our model also predicts exceptionally large isolated "spikes" in $z(t)$. These events are distinct phenomena associated with "avoided transitions" that occur when the system's trajectory $(r_\lambda(t),\ p_\lambda{}^*(t))$ crosses the relevant spinodal, stays in the "forbidden region" for a time typically shorter than the relaxation time of the system, and then returns to the hysteresis region. These sharp drops followed by rapid recovery are possibly related to the phenomenon of "flash crashes" seen in real-world dynamic networks. A notable example from economics is the 6 May 2010 "flash crash" of the US stock markets (see Supplementary Figure 1(a), below). On that day stock markets in the US experienced a rapid loss in market index value followed by a rapid recovery. Supplementary Figure 1(b) shows a sharp drop from Fig. 2(a) in the manuscript (the green circle that denotes the "avoided transition" at time $t \approx 5530$), which is magnified to see the structure of the event. The system's trajectory $(r_\lambda(t),\ p_\lambda{}^*(t))$ approached the "right" spinodal (decreasing the value of z below 0.6), and crossed it at the moment indicated by the red arrow. That moment was followed by a rapid drop and recovery that lasted only a few time steps, followed by a slower recovery as the system's trajectory returned to the hysteresis region and slowly left the vicinity of the spinodal.

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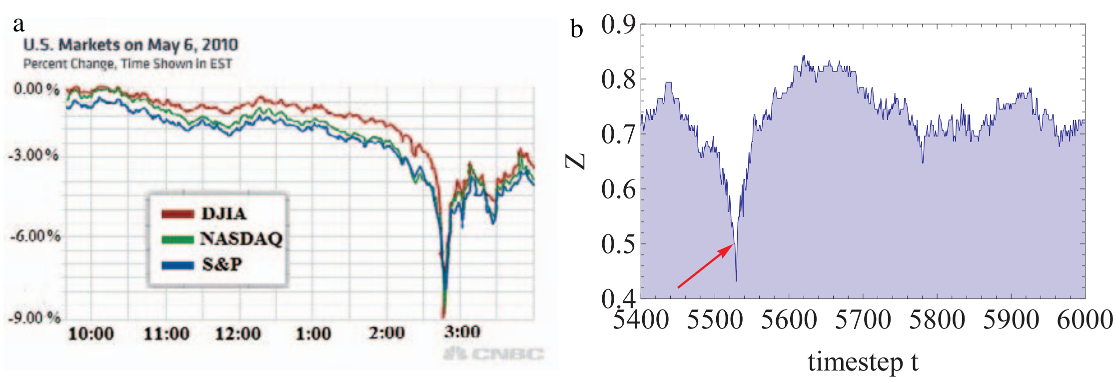

Supplementary Figure 1: *Flash crash.* (a) The infamous flash crash of the US financial market on 6 May 2010. The market index value dropped rapidly, followed by an equally rapid recovery. Source: *CNBC*. (b) Isolated sharp drop associated with an "avoided transition", in our model.