Microwave amplification with nanomechanical resonators:  
Supplementary Information

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1 Description of the experiment

1.1 Sample fabrication

In order to minimize the stray capacitance \( C \) of the cavity, the device was fabricated on a fused silica (SiO\(_2\), glass) substrate, which has a low dielectric constant (\( \epsilon_r \approx 4 \)), as compared to, for instance, silicon (\( \epsilon_r \approx 12 \)).

Both the cavity and the structures for the beam were patterned in a single e-beam lithography step, followed by evaporation of 150 nm aluminum. In order to suspend the beams, the substrate was etched by the use of HF vapor etcher, for 500 seconds at a pressure of 150 torr. The use of HF vapor instead of liquid oxide etchant is necessary in order to avoid damaging the aluminum film. The depth of the roughly isotropic etch was about 700 nm.

The mechanical beams were defined by Focused Ion Beam (FIB) etching, as in Ref. [1]. In order to create a uniform 9-12 nm vacuum slit over the whole length \( L = 8.5 \mu m \) of the beam,
we used low gallium ion currents of 1.5 pA and 75% exposure overlap in a single cutting pass.

In order to avoid charge accumulation due to the insulating substrate, all the structures were kept galvanically short-circuited and connected to ground until the very end of fabrication.

1.2 Cavity design and characterization

The cavity was designed and fabricated such that it would have a high critical current in order to enable a high drive $n_c \gtrsim 10^8$, and as small stray capacitance $C$ as possible. The first requirement suggests to fabricate it in a single lithography step. The low dielectric constant ($\epsilon_r \approx 4$) of the substrate contributes to a low stray capacitance, moreover, the roughly isotropic release etch for the beam, which partially suspended also the cavity, finally contributed such that $\epsilon_r \sim 3$ for the final structure.

The cavity design (Fig. 1a) is a 2 microns wide, 45 mm long meandering microstrip floating from both ends, and the mode we are using is the lowest mode of the structure which roughly corresponds to the $\lambda/2$ resonance in a typical transmission line resonator (where the cross-coupling between adjacent meanders is negligible). There are similar interdigital coupling capacitors $C_c \approx 6$ fF in both ends, however, only one of them is used, while the other one is shorted to the ground. In order to deduce the value of $C$, and the validity of the parallel $LC$ model in the somewhat complicated structure whatsoever, we made electromagnetic simulations with ideal inductor and capacitor components $C_g$ and $L_g$ inserted between the open ends of the meander, see Fig. 1a,b. By inspecting how their values affect the mode frequency, one can extract $C$ and $L$ from $\omega_c = \left[\left(\frac{L}{L_g}\right)(C + C_g)\right]^{-1/2}$ (ignoring the effects of $C_c$, $C_{S1}$ and $C_{S2}$ shown in Fig. 1a of the main text). The effective stray capacitance, which sets the coupling energy, is then roughly the parallel sum $C + C_c \approx 24$ fF.

The values of $C_S$ are further determined from a lumped element circuit simulation, by comparing to the measured reflection parameters. From the experiment, we obtain the FWHM of the
Figure 1: **Design for the cavity.** a, Simulation drawing for the meandering cavity structure, with ideal circuit components inserted between the open ends; b, change of the cavity resonance for 1 fF change of \( C_g \); c, micrograph showing the clamped beam and part of the cavity. The roughly isotropic etch causes about 700 nm undercut also for the cavity.

\( S_{11} \) of magnitude \((2\pi) \times 6.0\) MHz, and the maximum absorption at resonance is -5.5 dB. Note that there is no accurate simple relation for obtaining \( \gamma_c = \omega_c/Q_c \) from the reflection measures. Instead, it holds for the driven response \( \chi(\omega) \) of the \( LC \) circuit that FWHM \( \Delta\omega_c = \sqrt{3}\gamma_c \). We determine \( \Delta\omega_c \) from the lumped element simulation: internal losses, modeled by a resistor, are first adjusted to match the measured absorption. The driven response function, depending on \( \Delta\omega_c \), is then given by the frequency dependence of the current \( i_L \) flowing through \( L \),

\[
\chi(\omega) \equiv \frac{i_L(\omega)}{i_L(\omega_c)} = \frac{\gamma_c^2 \omega_c \omega}{\sqrt{\gamma_c^2 \omega^2 + (\omega^2 - \omega_c^2)^2}}.
\]

(1)

This way, we obtain \( \Delta\omega_c \approx (2\pi) \times 12.2\) MHz, and finally an estimate \( \gamma_{c,S_{11}} = (2\pi) \times 7.0\) MHz. The ratio of the internal and external dissipation \( \gamma_E/\gamma_I \approx 3.4 \) is determined by the resonance absorption.
The above results have rather large error bars. In section 1.4, we explain the more accurate pump detuning measurement, yielding the numbers $\gamma_c = (2\pi) \times 6.2$ MHz, $\gamma_I = (2\pi) \times 1.4$ MHz and $\gamma_E = (2\pi) \times 4.8$ MHz used in the rest of the manuscript and in a rough accordance with the values quoted above.

The number of quanta $n_c$ in the cavity at a given detuning is given by $n_c(\omega)/n_c(\omega_c) = \chi^2(\omega)$, and $n_c(\omega_c) = L i_L(\omega_c)^2$. The current response $i_L(\omega_c)$ at a given input power is again obtained from lumped element simulation.

### 1.3 Cryogenic setup

The experiments were carried out in a dilution refrigerator down to 25 mK temperatures. The pump and probe signals are combined at room temperature using a power splitter. Before the signals enter the cryostat, a sharp high-pass filter, see Fig. 2b, at room temperature is used to cut the phase noise of the generators near the cavity frequency. This filter provides 50 dB more attenuation at the cavity frequency than at the blue sideband. Without proper filtering, the phase noise would reflect from the cavity, and appear as extra added noise of tens of quanta. Inside the cryostat, the incoming signals are attenuated by $43 \pm 1.5$ dB. The uncertainty in the cryogenic attenuation sets relatively large error bars for $n_c$. Thermal noise emanating from higher temperatures is estimated to contribute less than 0.1 quanta of thermal occupancy into the cavity, and is thus a negligible contribution to the total noise. The entire setup is described in Fig. 2a.

The signals reflected from the amplifier chip are directed to the cryogenic amplifier which has a high input compression point of -20 dBm, allowing to use high pump powers without problems of amplifier saturation. The amplifier has a noise temperature $\sim 4$ K. In addition, there is attenuation of 2...2.5 dB due to circulators and cables between the sample and the amplifier. The effective noise temperature, which sets the signal-to-noise ratio, is then 6...7 K.
Figure 2: Setup for electronics and the microwave cabling. a, Inside the dilution refrigerator, we use beryllium copper (BeCu), copper nickel (CuNi), and niobium titanium (NbTi) coaxial cables. Inner-pin DC-blocks (DCB), and high-pass filters (HP) are used to reduce heat leak. Back at room temperature, the pump is blocked from the output signal. After further amplification, the signal microwave is recorded coherently in a network analyzer. 

b, Measured attenuation of the input filter (tunable 7-8 GHz cavity bandpass filter WBCQV7000/8000-6SSSD from Wainwright Instruments) used to purify carrier tone.
1.4 Characterization of the electromechanical system

In order to establish a good understanding of the basic behavior of the electromechanical system, we determined its parameters independently from the amplification measurements.

For determining the electromechanical coupling energy $g = \frac{w_0}{2C} \frac{\partial C}{\partial x}$, we used the value for $C \simeq 24 \text{ fF}$ as obtained in Sec. 1.2. Moreover, $\frac{\partial C}{\partial x} \simeq 13 \text{ nF/m}$ is estimated from the dimensions of the beam and the vacuum slit. We get $g = (2\pi) \times 1.8 \text{ MHz/nm}$, which corresponds to the shift of the cavity frequency of 40 Hz per phonon. Similarly as previously done in Refs. [2, 3], we made measurements where the pump frequency or power is varied near the blue sideband. This alters the optical spring effect which can be compared to the theory for shifts of the frequency and damping, Eqs. (26), (27) below. The effective mechanical frequency can be read from the position of the mechanical sideband, more precisely, from the departure of this peak from the pump frequency.

Figure 3: Characterization of the electromechanical system via the optical spring effect. The incident microwave power was kept constant such that at the blue sideband frequency, $n_c \sim 5.3 \times 10^5$.

In Fig. 3 we compare the measured effective mechanical frequency to the theory. Corresponding plot for the damping is shown in Fig. 2c of the main text. The best fit is obtained with $\gamma_c = 6.2 \text{ MHz}$. This value differs by 10% from that deduced from the $S_{11}$ measurement. We
consider this value of $\gamma_c$ the most reliable, and use it in the rest of the paper.

The values of $n_c$ we get from these fits are about 30\% smaller than those from independent estimates based on the input attenuation and cavity response. We attribute this difference to the somewhat inaccurately known cryogenic attenuation, which has a sensitive effect on $n_c$. We adjust the scale of $n_c$ according to these fits, and quote the adjusted values in the paper. For instance, a useful fixed point is the instability point, which is expected according to theory at $n_c \sim 1.2 \times 10^6$ in the situation of Fig. 3b in the main text.

1.5 Determination of the noise added by the mechanical amplifier

The noise temperature of an amplifier is determined by comparing its noise to a known noise source. Here, the noise floor which establishes the signal-to-noise ratio is set by the effective noise temperature of the system, approximately 6...7 K.

We worked at a temperature of 30 mK, and used a weak input signal as a marker, see Fig. 4. The marker peak height versus noise floor is improved by 2.3 dB by the mechanical amplification, however, this has to be subtracted by the cavity absorption (here, -1.7 dB). We thus obtain a slight 0.6 dB improvement to the signal-to-noise ratio, which corresponds to 20 added noise quanta, matching the expectation equaling the thermal phonon number.

2 Theoretical details

2.1 Quantum Langevin equation for the optomechanical system

In this section we derive the dynamical equations for the cavity and the mechanical degrees of freedom for our system. After defining the Hamiltonians describing the two oscillators and the parametric coupling, we write the (non-linear) Hamilton’s equations for the system. Following a standard dynamical-system approach, we separate the dynamical variables into stationary values (in the proper rotating frame) and corresponding fluctuations. In particular, the solutions
Figure 4: **Added noise of the mechanical amplifier** a. A weak probe signal (narrow peak) is employed in order to deduce the signal-to-noise ratio with the amplification off (black), or on (blue). $\Delta \simeq 0.89\omega_m$, temperature $T = 30$ mK. The bump about the probe peak is due to the thermomechanical vibrations; b. Theoretical plot of the added noise at the optimal value of the effective coupling for different values of $n_m$ and $n_c$, using the same device parameters as in the experiment. The deviation between the quantum limit and the case with $n_m = n_c = 0$ is due to the presence of a finite $\gamma_I/\gamma_E$.

of the dynamical equations for the stationary values allow to determine the value of the cavity field as a function of the pump field. These solutions set the “operating point” of the amplifier,
fixing the values of the effective parameters for the fluctuations dynamics.

We now explicitly derive the quantum Langevin equations (QLE) for the optomechanical system. In the absence of any coupling to the external world, the system (oscillator+cavity) Hamiltonian can be written as

\[ H_{\text{sys}} = \hbar \omega_c \left( a_T^\dagger a_T + \frac{1}{2} \right) + H_{\text{ho}} + H_{\text{int}}, \]  

where \( a_T \) is the cavity field operator, and \( \omega_c \) the cavity resonant frequency,

\[ H_{\text{ho}} = \frac{p_T^2}{2m} + \frac{1}{2} m \omega_m x^2 \]

is the mechanical harmonic oscillator Hamiltonian, \( m \) the mass of the mechanical system and \( \omega_m \) its resonant frequency. The Hamiltonian

\[ H_{\text{int}} = -\hbar g \left( a_T^\dagger a_T + \frac{1}{2} \right) x \]  

is the parametric interaction Hamiltonian, where \( g \) is the coupling between the mechanical degrees of freedom and the cavity. The Hamiltonian coupling the cavity with the external radiation modes can be written as

\[ H^{(I,E)}_{\text{rc}} = i\hbar \int_{-\infty}^{\infty} d\omega s_{(I,E)}(\omega) \left[ b_{(I,E)}^\dagger(\omega) a_T - b_{(I,E)}(\omega) a_T^\dagger \right], \]

where \( s_{(I,E)} \) describes the cavity/reservoir couplings, and the indeces \( I, E \) refer to the external and internal baths, respectively. The external bath is associated with the transmission line coupling the input and output signal with the cavity, while the internal ones refer to any other source of dissipation potentially coupling to the cavity.

The reservoir associated with the dissipative dynamics of the mechanical oscillator (below, the mechanical bath) can be written as

\[ H_{\text{mech}} = \frac{1}{2} \sum_j \left[ (p_j - k_j x)^2 + \omega_j^2 q_j^2 \right]. \]
$H_{\text{mech}}$ corresponds thus to describing the reservoir in terms of a collection of independent harmonic oscillators with frequencies $\omega_j$, each of which is coupled to the mechanical oscillator through $k_j$ [4].

With the aid of the input-output formalism [5], the evolution equations for the cavity field operators and the position and momentum operators for the mechanical system can be written as

\begin{align}
\dot{x} &= \frac{p_T}{m} \\
\dot{p}_T &= -m\omega_m^2 x + \hbar g \left( a_T^\dagger a_T + \frac{1}{2} \right) - \gamma_m p_T + \xi_T \\
\dot{a}_T &= -i (\omega_c - \omega_p - g x) a_T - \frac{\gamma_c}{2} a_T - \sqrt{\gamma_I} a_T^\dagger a_{\text{in}}^\dagger - \sqrt{\gamma_E} a_T a_{\text{in}}. 
\end{align}

Here we have made the rotating wave approximation for the cavity field. Thus, we consider a situation where the cavity is strongly driven by a coherent field oscillating at frequency $\omega_p$. Moreover $\gamma_E$, $\gamma_I$ represent the losses associated with the input/output port and the photon bath associated with the internal losses of the cavity ($\gamma_c = \gamma_I + \gamma_E$), and $\gamma_m$ the mechanical losses.

We now linearize Eqs. (8-7), rewriting $a_T$, $x$ and $p_T$ as the sum of a coherent field and a quantum operator

\begin{align}
a_T &= \alpha + a \\
x &= \chi + \sqrt{\frac{\hbar}{m\omega_m}} q \\
p_T &= \pi + \sqrt{\hbar m\omega_m} p. 
\end{align}

More specifically, we have rewritten Eq. (9) with a view to the decomposition in terms of a (coherent) pump field $\alpha_p$ and input signal and noise sources, i.e.,

$$a_{T\text{in}} = \alpha_p + a_{\text{in}}$$

Since we are first interested in the steady-state solution, neglecting all fluctuations, we impose the condition $\dot{\alpha} = \dot{\chi} = \dot{\pi} = 0$, leading to the steady-state values (in a frame rotating at $\omega_p$,
denoted with subscript s)

\[ \pi_s = 0 \]  \hspace{1cm} (12)

\[ \chi_s = \frac{\hbar g}{m\omega^2_m} \left( |\alpha_s|^2 + \frac{1}{2} \right) \]  \hspace{1cm} (13)

\[ \alpha_s = \frac{\sqrt{\gamma_E \alpha_p}}{2} + i \left( \omega_c - \omega_p - g\chi_s \right). \]  \hspace{1cm} (14)

Equations (12, 14) can be combined into a third-order algebraic equation, leading to three stationary solutions or the cavity field \( \alpha_s \) as a function of the pump field \( \alpha_p \).

We now focus on the solution for which \( \alpha_s \to 0 \) when \( \alpha_p \to 0 \). In this case, the evolution equations for the fluctuation operators can be written as

\[ \dot{q} = \omega_m p \]  \hspace{1cm} (15)

\[ \dot{p} = -\omega_m q - \gamma_m p + \frac{\tilde{g}}{\sqrt{2}} \left( a^\dagger + a \right) + \xi \]  \hspace{1cm} (16)

\[ \dot{a} = i\Delta a - \frac{\gamma_c}{2} a + \frac{i\tilde{g}}{\sqrt{2}} q + \sum_{i=I,E} \sqrt{\gamma_i} a^i_{in}, \]  \hspace{1cm} (17)

where \( \Delta = \omega_p - \omega_c - g\chi_s \) and \( \tilde{g} = 2g\sqrt{\frac{\hbar}{2m_{\text{m}}}} \alpha_s \). We also assumed, without loss of generality, that \( \alpha_s \) is real. Equations (15-17) represent the quantum Langevin equations for the cavity+mechanical resonator system. It is worth noting here that, following [6], we have not performed the rotating wave approximation for the mechanical bath degrees of freedom. This choice affects the noise spectrum for the operator \( \xi \).

**2.2 Amplification, squeezing**

The input and output fields at the input/output port of the cavity are related by [5]

\[ a_{out} = \sqrt{\gamma_E} a + a_{in}, \]  \hspace{1cm} (18)
where \( a \) is solved from Eqs. (15-17). This leads to the general relation between the output field and the various incoming fields,

\[
a_{\text{out}}(\omega) = M(\omega)a_{\text{in}}(\omega) + L(\omega)a_{\text{in}}^\dagger(\omega) + M_I(\omega)a_{\text{in}}(\omega) + L_I(\omega)a_{\text{in}}^\dagger(\omega) + Q(\omega)\xi(\omega),
\]

where \( a_{\text{in}}^\dagger(\omega) \) and \( \xi(\omega) \) represent the noise introduced by the internal losses of the cavity and the mechanical bath (see Fig. 5). The amplitude gains for the input signal \((M, L)\), and those for the input noise \((M_I, L_I, Q)\) are given by

\[
\begin{align*}
M(\omega) &= \left[ \Gamma_M(\omega) - \frac{i\gamma_E(\Gamma_M(\omega) - 1)}{2\Delta} - 1 \right] \\
L(\omega) &= -i\frac{\gamma_E(\Gamma_M(\omega) - 1)}{2\Delta} \\
M_I(\omega) &= \left[ \Gamma_M(\omega) - \frac{\sqrt{\gamma_E\gamma_I}}{\gamma_c/2 - i(\omega + \Delta)} - i\frac{\gamma_E\gamma_I(\Gamma_M(\omega) - 1)}{2\Delta} \right] \\
L_I(\omega) &= -i\frac{\sqrt{\gamma_E\gamma_I}(\Gamma_M(\omega) - 1)}{2\Delta} \\
Q(\omega) &= \sqrt{\frac{\gamma_c}{2}} \frac{\Gamma_M(\omega) - 1}{\Delta\tilde{g}} \left[ (\Delta - \omega) - \frac{i\gamma_c}{2} \right].
\end{align*}
\]

The key role in the amplification is played by the factor

\[
\Gamma_M(\omega) = \frac{\omega_m^2 - \omega^2 - i\gamma_m\omega}{\omega eff^2 - \omega^2 - i\gamma eff\omega},
\]

which, in turn, depends on the effective resonant frequency

\[
\omega eff = \left[ \frac{\omega_m^2 + \tilde{g}^2\Delta\omega_m}{\gamma_c^2/4 + (\omega - \Delta)^2} \left( \frac{\gamma_c^2/4 + \omega^2 + \Delta^2}{\gamma_c^2/4 + (\omega + \Delta)^2} \right) \right]^{1/2}
\]

and the effective damping coefficient

\[
\gamma eff = \left[ \frac{\gamma_m - \tilde{g}^2\Delta\omega_m}{\gamma_c^2/4 + (\omega - \Delta)^2} \left( \frac{\gamma_c^2/4 + \omega^2 + \Delta^2}{\gamma_c^2/4 + (\omega + \Delta)^2} \right) \right],
\]

induced by the coupling of the mechanical resonator with the cavity. At resonance \( \omega \approx \omega_m \) and neglecting the (weak) pump dependence of \( \omega eff \), it is clear that a decrease of \( \gamma eff \to 0 \) leads to
$\Gamma M \gg 1$ (see below for stability considerations), and thus, through the $\Gamma M$ dependence of $M$ and $L$, to an amplification of an input signal.

The explicit form of $L(\omega)$ given by Eq. (21) implies that the origin of the squeezing effect ($L(\omega) \neq 0$) is associated to an off-resonant process, mediated by the mechanical oscillator. In the good cavity limit, the off-resonant process becomes irrelevant, thus preventing the possibility of observing squeezing in this case. This situation is opposite to the findings in studies on two-tone back-action evading measurements [7], where the strongest squeezing is found in the good-cavity limit.

We use the notion of preferred quadratures [8], for which the output quadrature fields $X_{\text{out}}$, $Y_{\text{out}}$ are independent of $Y_{\text{in}}$ and $X_{\text{in}}$, respectively. The power gains in these quadratures are obtained as

$$G_x = (|M| + |L|)^2 \quad (28)$$
$$G_y = (|M| - |L|)^2 \quad (29)$$
$$G_{\text{av}} = \frac{1}{2} (G_x + G_y) = |M|^2 + |L|^2 \quad . \quad (30)$$

The expression of the gain in the preferred quadratures corresponds to the possibility of choosing an appropriate phase for $a_{\text{in}}$ and $a_{\text{out}}$ such that they lead to real-valued expressions for $M$ and $L$, given by Eqs. (20, 21). In these quadratures, the amplifier equations can be written as (dropping here the added-noise terms)

$$X_{\text{out}} = (|M| + |L|) X_{\text{in}} \quad (31)$$
$$Y_{\text{out}} = (|M| - |L|) Y_{\text{in}} \quad , \quad (32)$$

thus leading to the relations given by Eqs. (28, 29) for $G_x$ and $G_y$. In addition to the trivial difference associated with the condition $\gamma_E \neq \gamma_I$, reflecting different coupling mechanisms of the cavity to the external world, the expressions for $M_I(\omega)$ and $M(\omega)$ differ due to the interference
This term represents the interference between the input (either signal or noise) that has been reflected at the cavity/transmission line interface and the input that has travelled through the cavity. The expression of the gains given by Eqs. (28-30) involve only the coefficients $M(\omega)$ and $L(\omega)$, since the signal is supposed to enter the system only through the input external port.

On the other hand, while opening the cavity to the transmission line also opens that port to the noise from the transmission line, this noise is regarded as intrinsic noise of the input signal and thus does not contribute to the noise added by the amplifier. The system is thus open to one signal source (the coherent part of $a_{in}$), the noise associated with internal losses ($a_{in}^I$), the noise associated with the mechanical bath ($\xi$) and the noise from the transmission line (the incoherent part of $a_{in}$), the latter not contributing to the noise added by the amplifier.

### 2.3 Noise: input field correlators and the quantum limit

Within our scheme, the noise added by the amplifier can be expressed in terms of noise spectra associated with the internal losses and the mechanical bath. Following a standard approach [9], the correlators for $a_{in}$ and $a_{in}^I$ are given by

$$\langle a_{in}^{(I)}(t)a_{in}^{(I)}(t')\rangle = [n(\omega_c) + 1]\delta(t - t')$$

(33)

$$\langle a_{in}^{(I)\dagger}(t)a_{in}^{(I)}(t')\rangle = n(\omega_c)\delta(t - t') .$$

(34)

Similarly, the mechanical noise correlator can be written as [10]

$$\langle \xi(t)\xi(t')\rangle = \int \frac{d\omega}{2\pi} \exp[-i\omega(t - t')] S_\xi(\omega) ,$$

(35)

with

$$S_\xi(\omega) = 2\gamma_m \frac{\omega}{\omega_m} \left\{ (n_\omega + 1)\Theta(\omega) + [(n_{-\omega} + 1)\Theta(-\omega)] \right\} ,$$

(36)
Figure 5: Schematics of the amplification scheme with an outline of the different noise sources. $a_{\text{in}} = \alpha_{\text{in}} + \delta a_{\text{in}}$ is the input field associated with the input signal $\alpha_{\text{in}}$ and the noise at the input port $\delta a_{\text{in}}$, $a_{\text{in}}'$ is the field associated with the noise reservoir acting directly on the resonant cavity and $\xi$ is the mechanical noise associated with the thermal bath.

where $\Theta(x)$ is the Heaviside step function (see e.g. [10]). Considering a thermally populated bath, the noise spectrum assumes the form [9]

$$S_\xi(\omega) = \gamma_m \frac{\omega}{\omega_m} \left[ \coth \left( \frac{\hbar \omega}{kT} \right) + 1 \right].$$

(37)

We are now in a position to evaluate the noise added by the amplifier. We define the operators

$$F_x = \frac{1}{\sqrt{2}} \left[ M_I a_{\text{in}}^\dagger + L_I a_{\text{in}}^\dagger + Q \xi + \text{h.c.} \right]$$

(38)

$$F_y = -i \frac{1}{\sqrt{2}} \left[ M_I a_{\text{in}}^\dagger + L_I a_{\text{in}}^\dagger + Q \xi - \text{h.c.} \right],$$

(39)

where the appropriate phase has been included in the definition of $M_I$, $L_I$ and $Q$ in order to satisfy the condition $M, L \in \mathbb{R}$. $(\Delta F_x)^2$ and $(\Delta F_y)^2$ represent the added noise by the amplifier [8]. The condition establishing a lower bound for the added noise reads in this case

$$\sqrt{|\Delta F_x|^2 |\Delta F_y|^2} / (G_x G_y) \geq \frac{1}{2} \left| 1 - (G_x G_y)^{-1/2} \right|. \quad (40)$$
Close to the optimal effective coupling \( g_{\text{opt}} = \sqrt{\gamma_m \gamma_c} \), and for \( \omega \approx \omega_m \), the expression for the added noise is given by

\[
\sqrt{|\Delta F_x|^2 |\Delta F_y|^2 / (G_x G_y)} \simeq \gamma_m \left(n_{\text{opt}} + 1/2\right) + \frac{\gamma_c}{\gamma_E} \left(n_m + 1/2\right).
\] (41)

From Eq. (41) it is possible to see that the quantum limit for the amplification can be reached in the absence of internal cavity losses and for a zero-temperature mechanical reservoir. In our experimental setup \( n_{\text{opt}} \simeq 0 \), leading to a linear increase of the added noise with the number of mechanical reservoir phonons. The prefactor of this linear dependence is given by the ratio between total and external losses.

### 2.4 Stability and validity of the linearized model

In obtaining the QLE for the cavity and the mechanical resonator, we have linearized the equations of motion for the cavity+mechanical resonator system, see Eqs. (9)-(11). Here we discuss the criterion for the stability and the limits of validity of the (linear) QLE considered to analyze the system dynamics (Ginsburg criterion) [11]. The requirement for the system stability is that the poles of the effective mechanical susceptibility, induced by the coupling between the mechanical resonator and the cavity, lie in the lower complex half-plane. In other words, the effective mechanical damping \( \gamma_{\text{eff}} \) must be positive in order for the system to be stable. The condition \( \gamma_{\text{eff}} \rightarrow 0^+ \) corresponds to the situation of maximal gain and, on crossing the \( \gamma_{\text{eff}} = 0 \) value, to the loss of stability. In the linearization procedure we assume that the term \( x \cdot a_T \) appearing in Eq. (8) can be expanded as

\[
\left(\xi + q\right) \cdot \left(\alpha + a\right) \simeq \xi\alpha + \alpha q + \xi a,
\] (42)

and analogously,

\[
a_T^\dagger a_T \simeq \alpha^2 + \alpha^\ast a + \alpha a^\dagger.
\] (43)
Equations (42) and (43) thus establish that, for the linearized QLE equations to aptly describe the dynamics of the optomechanical system, the following conditions must be met

\[ \frac{\langle qa \rangle}{\alpha_s \chi_s} \ll 1 \]  
\[ \frac{\langle a^\dagger a \rangle}{\alpha_s^2} \ll 1. \]  

(44)  
(45)

The solutions for \( q \) and \( a \) of the QLE as a function of the input field \( a_{in} \) lead to the following condition for the ratio between the signal and the pump power

\[ \frac{\langle a_{in}^\dagger a_{in} \rangle}{\alpha_s^2} \ll \frac{1}{|\Gamma_M|^2}, \]  

(46)

i.e., the signal amplitude after amplification must have a lower amplitude that that of the pump. It is thus clear that for large enough values of \( |\Gamma_M| \), the linearized description of the system

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Figure 6: Number of signal photons ensuring the validity of the linear-regime analysis as a function of \( \omega \) and \( G \). We have assumed \( \left( \frac{\langle a_{in}^\dagger a_{in} \rangle}{\alpha_s^2} \right)_{\text{threshold}} = 10^{-4}/|\Gamma_M|^2 \). The parameters have been chosen in order to match the values measured in the experiment.
physics breaks down. However as it can be seen from Fig. 6, there is a large range of parameters where, while having a gain significantly larger than 1, the linear model is still valid.

2.5 Future prospects

Let us consider the prospects to reach nearly quantum-limited operation of the device in the phase-insensitive mode. Let us first suppose essentially the same setup as presently, but with a short beam only 0.8 micron long and 50 nm thick and wide. This beam has $\omega_m/2\pi \sim 500$ MHz, and is nearly in the ground state at 20 mK, and thus $n_{\text{add}} \lesssim 1$, on par with the best Josephson devices. We should take a larger $\gamma_E \sim 50$ MHz by increasing $C_e$. Then, we have $g/2\pi = 10$ Hz. The effect of increasing mechanical frequency does not affect the values of gain or noise, but the somewhat decreased coupling strength must be compensated by an equally increased pump amplitude. Here, to reach the maximum gain, one would need $n_{c, \text{crit}} \sim 10^8$ which was still possible in our experiment, but might be prone to cause heating. With a narrower gap of 5 nm, which is possible to accomplish in shorter beams, we have $n_{c, \text{crit}} \sim 10^7$, near the present value.

Another possible approach towards the quantum limit might be pre-cooling of the mechanics near the ground state [12] by inverting the pump to the red sideband. We note that the cooling tone cannot be applied simultaneously to the amplification pump since they would cancel each other. Hence, the setup should be pulsed: applying a cooling pump pulse followed by an amplification pump pulse might allow to reduce the noise emanating from the mechanical resonator. However, detailed analysis of this would require considering transient effects.

As our amplifier scheme containing a coupling of two oscillators does not explicitely require superconductivity, it could be realized for example in the much sought-after THz regime as well by fabricating a smaller cavity with a resonance frequency within this regime.
References


